A LATTICE ON DYCK PATHS CLOSE TO THE TAMARI LATTICE

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Abstract. We introduce a new poset structure on Dyck paths where the covering relation is a particular case of the relation inducing the Tamari lattice. We prove that the transitive closure of this relation endows Dyck paths with a lattice structure. We provide a trivariate generating function counting the number of Dyck paths with respect to the semilength, the numbers of outgoing and incoming edges in the Hasse diagram. We deduce the numbers of coverings, meet and join irreducible elements. We give a generating function for the number of intervals, and we compare this number with the number of intervals in the Tamari lattice. Finally, we present a sequent calculus capturing this new Lattice.

In this paper, we introduce a new poset structure on Dyck paths where the covering relation is a particular case of the covering relation that generates the Tamari lattice [3, 5]. A Dyck path is a lattice path in $\mathbb{N}^2$ starting at the origin, ending on the x-axis and consisting of up steps $U = (1, 1)$ and down steps $D = (1, -1)$. Let $\mathcal{D}_n$ be the set of Dyck paths of semilength n (i.e., with $2n$ steps), and $\mathcal{D} = \cup_{n\geq 0}\mathcal{D}_n$. The cardinality of $\mathcal{D}_n$ is the n-th Catalan number $c_n = (2n)!/(n!(n + 1)!)$ (see A000108 [1]).

A *peak* in a Dyck path is an occurrence of the subpath UD. A *pyramid* is a maximal occurrence of U^kD^ℓ, $k, \ell \geq 1$, in the sense that this occurrence cannot be extended in a occurrence of $U^{k+1}D^\ell$ or in a occurrence of $U^kD^{\ell+1}$. We say that a pyramid U^kD^ℓ is *symmetric* whenever $k = \ell$, and *asymmetric* otherwise. The *weight* of a symmetric pyramid U^kD^k is k. For instance, the Dyck path in the south west of Figure 1b contains two symmetric pyramids and two asymmetric pyramids.

In 1962 [5], the Tamari lattice $\mathcal{T}_n$ of order n is defined by endowing the set $\mathcal{D}_n$ with the transitive closure $\preceq$ of the covering relation $P \xrightarrow{\mathcal{T}} P'$ that transforms an occurrence of $DUQD$ in P into an occurrence $UQDD$ in P', where Q is a Dyck path (possibly empty). The top part of Figure 1b shows an example of such a covering, and Figure 1a illustrates the Hasse diagram of $\mathcal{T}_n$ for $n = 4$ (the red edge must be considered).

Now, we introduce a new partial order on $\mathcal{D}_n$ for $n \geq 0$. We endow it with the ordered relation $\leq$ defined by the transitive closure of the following covering relation $P \longrightarrow P'$ that transforms an occurrence of DU^kD^k in P into an occurrence U^kD^kD in P', where $k \geq 1$. For short, we will often use the notation $DU^kD^k \longrightarrow U^kD^kD,\ k \geq 1$ whenever we need to show where the transformation is applied.

Notice that the covering $\longrightarrow$ is a particular case of the Tamari covering $\xrightarrow{\mathcal{T}}$ whenever we take $Q = U^{k-1}D^{k-1}$ in the transformation $DUQD \xrightarrow{\mathcal{T}} UQDD$. Let $\mathcal{S}_n$ be the poset $(\mathcal{D}_n, \leq)$. The bottom part of Figure 1b shows an example of such a covering, and Figure 1a

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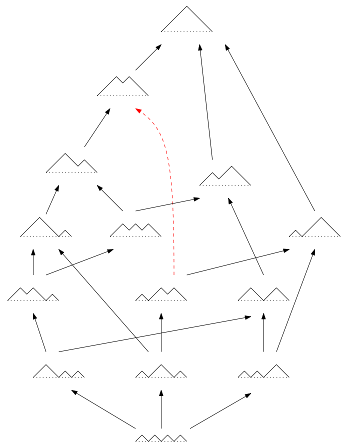
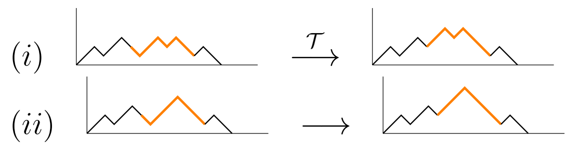
[1]All sequences referenced in this paper can be found in [4]



(A) The Hasse diagram of $\mathcal{S}_4 = (\mathcal{D}_4, \leq)$. The Tamari lattice $\mathcal{T}_4 = (\mathcal{D}_4, \preceq)$ can be viewed by considering the dotted edge (in red).



(B) (i) corresponds to a covering relation for the Tamari Lattice, while (ii) corresponds to the covering relation for the new lattice of this study.

illustrates the Hasse diagram of $\mathcal{S}_n$ for $n = 4$ (without the red edges which belongs to the Tamari lattice but not to this new poset).

Theorem 1. *The poset* $(\mathcal{D}_n, \leq)$ *is a lattice.*

Theorem 2. *We have*

$$A(x,y,z) = \frac{R(x,y,z) - \sqrt{4\,x\,(xzy - xy - xz + 1)\,(xy + xz - x - 1) + R(x,y,z)^2}}{2x\,(xzy - xy - xz + 1)},$$

where $R(x,y,z) = x^2zy - x^2y - x^2z + x^2 - xy - xz + x + 1$.

Corollary 1. *The generating function $E(x)$ where the coefficient of x^n is the total number of possible coverings over all Dyck paths of semilength n (or equivalently the number of edges in the Hasse diagram) is*

$$E(x) = \frac{-1 + 4x + (1-2x)\sqrt{1-4x}}{2\,(1-4x)\,(1-x)}.$$

The coefficient of x^n is given by $\sum_{k=0}^{n-2}\binom{2k+2}{k}$. The ratio between the numbers of coverings in $\mathcal{T}_n$ and $\mathcal{S}_n$ tends towards $3/2$.

Let $K(x)$ be the generating function where the coefficient of x^n is the number of meet irreducible elements (Dyck paths with only one outgoing edge). Using the symmetry $y \longleftrightarrow z$ in $A(x,y,z)$, $K(x)$ also is the generating function where the coefficient of x^n is the number of join irreducible elements (Dyck paths with only one incoming edge).

Corollary 2. *We have* $K(x) = \frac{x^2}{(x-1)(2x-1)}$, *and the coefficient of x^n is* $2^{n-1} - 1$ *for* $n \geq 1$.

Denote by $\mathcal{L}$ the set of paths with only one outgoing edge and only one incoming edge. Let $L(x)$ be the generating function where the coefficient of x^n is the number of such paths.

Corollary 3. *We have* $L(x) = 3x^3 + \frac{(x+2)x^4}{1-x-x^2}$*, and the coefficient of* x^n *is* 0 *whenever* $n \leq 2$*, is* 3 *whenever* $n = 3$*, and is the Fibonacci number* F_{n-1} *otherwise, where* F_n *is defined by* $F_n = F_{n-1} + F_{n-2}$ *with* $F_1 = F_2 = 1$*.*

Now, we provide the generating function and a close form for the number of intervals in the lattice $\mathcal{S}_n$. The method is inspired by the work of Bousquet-Mélou and Chapoton [1]. We introduce the bivariate generating function $I(x,y) = \sum_{n,k\geq 1} a_{n,k} x^n y^k$, where $a_{n,k}$ corresponds to the number of intervals in $\mathcal{S}_n$ such that the upper path ends with k down-steps exactly. We also define $J(x,y) = \sum_{n,k\geq 1} b_{n,k} x^n y^k$, where $b_{n,k}$ corresponds to the number of intervals in $\mathcal{S}_n$ such that the upper path is prime and ends with k down-steps exactly (recall that a Dyck path is prime whenever it only touches the x-axis at its beginning and its end).

Theorem 3. *The generating function* $J(x,y)$ *for the number of intervals* (P,Q) *where* Q *is prime, with respect to the semilength and the size of the last run of down steps is*

$$J(x,y) = \frac{xy\left(-1 + J(x,1)\right)\left(J(x,1)\, C\left(xy\right) y - y + 1\right)}{J(x,1)\, C\left(xy\right) xy^2 - C\left(xy\right) xy^2 - xy^2 - J(x,1)\, y + xy + J(x,1) + y - 1}$$

with $J(x,1) = \frac{1-\sqrt{1-8x}}{4}$*, and* $C(x)$ *is the g.f. for Catalan numbers, i.e.* $C(x) = 1 + xC(x)^2$*.*

The sequence of coefficients corresponds to the sequence A052701 that counts outerplanar maps with a given number of edges [2]. The n-th coefficient is given by the close form $2^{n-1}c_{n-1}$, where $c_n = (2n)!/(n!(n+1)!)$ is the n-th Catalan number A000108.

Theorem 4. *The generating function* $I(x,y)$ *for the number of intervals* (P,Q) *with respect to the semilength and the size of the last run of down steps is* $I(x,y) = J(x,y) \cdot \frac{3-\sqrt{1-8x}}{2(x+1)}$*, and the generating function* $I(x,1)$ *for the number of intervals* (P,Q) *with respect to the semilength is*

$$I(x,1) = \frac{1 - 2\,x - \sqrt{1-8x}}{2(x+1)}.$$

where the sequence of coefficients corresponds to the sequence A064062 that counts simple outerplanar maps with a given number of vertices [2]. The n-th coefficient is given by the close form $\frac{1}{n}\sum_{m=0}^{n-1}(n-m)\binom{n+m-1}{m}2^m$.

From a more logical point of view we will see how to find a sequent calculus inspired from Lambek's product rule and adapted to the Tamari Lattice in [6] that captures the structure of this lattice and present our current results on it.

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