The Tamari order for $D^3$ and derivability in semi-associative Lambek-Grishin Calculus

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A $k$-dimensional Dyck language, $D^k$, consists of words over a $k$-letter alphabet (lexicographically ordered) satisfying the following two conditions:

- **MULTIPLICITY**: each word contains the $k$ letters with equal frequency
- **PREFIX**: for every prefix of a word, the number of $a_1 \geq$ the number of $a_2 \geq \ldots \geq$ the number of $a_k$

The familiar language of balanced brackets is $D^2$, with alphabet \{a, b\}, reading a, b as opening and closing bracket respectively. We write $D^k_n$ for the finite $D^k$ language with letter multiplicity $n$. For the place of $D^k$ in the extended Chomsky hierarchy, the conjecture is that $D^k$ languages are recognizable by $(k - 1)$-MCFG (multiple context-free grammars, [7]). $D^2$ is recognized by a 1-MCFG, in other words, a simple CFG.

I report on work in progress on extending the Tamari order for $D^2$ to the three-dimensional case, and on modelling the Tamari order for $D^3$ in terms of derivability in an extended version of Lambek’s Syntactic Calculus [4], going back to V.N. Grishin [2].

In [6], it is shown how words of $D^k_n$ can be interpreted as the Yamanouchi words of rectangular $k \times n$ standard Young tableaux. For the three-dimensional case, Borie [1] discusses a bijection between 3-row, $n$-column rectangular Young tableaux and $PC(n)$, single-input single-output product coproduct prographs with $n$ product (hence also $n$ coproduct) nodes.

The bijection is obtained by means of a depth-left first traversal of a $PC$ graph with the condition that the output edge of a product node can be visited only if its two incoming edges have already been visited. Inputs of coproducts provide the entries of the top row of the corresponding tableau; left inputs of products give the entries of the middle row; right inputs of products of the bottom row. (For the Yamanouchi word over alphabet \{a, b, c\} associated with a tableau, the top row entries give the positions where letter $a$ appears, the middle row letter $b$, the bottom row letter $c$.) As an illustration, on the left of Figure 1, the Young tableau with Yamanouchi word $aabbacbcc \in D^3_3$ and the corresponding $PC(3)$ graph with the steps of its depth-left first traversal.

An order on the $PC(n)$ graphs, extending the two-dimensional Tamari order discussed in [9] to three dimensions can be obtained by means of the local transformations below: (a) $\alpha^\Delta$ coproduct semi-associativity (cf left rotation in [9]), (b) $\alpha_\mu$ product semi-associativity (dual to the above, 180° rotation), (c) $\alpha^\Delta_\mu$ mixed product-coproduct semi-associativity. The transformations (a)-(c) induce a lattice for the $D^3_n$ order with $(abc)^n$ as minimum, and $a^n b^n c^n$ as maximum.

As an example, the graph for the word $aabbacbcc$ in Fig 1 matches the input conditions for coproduct semi-associativity (marked red) and for product semi-associativity (blue). The results of rewriting with the associated Young tableau and its Yamanouchi word are shown in the middle and on the right.

Zeilberger [9] shows that $A \leq B$ in the Tamari order for $D^2$ can be modelled as logical derivability $A \vdash B$ in a variant of Lambek’s Syntactic Calculus with a restricted form of associativity. I investigate...
Figure 1: Graph rewriting: $aabbacbcc \xrightarrow{\alpha} aaabbcbcc$ (red), $aabbacbcc \xrightarrow{\alpha} aabbabccc$ (blue)

whether a similar logical perspective on the $D^3$ order can be obtained in a suitably restricted form of the Lambek-Grishin calculus $LG$ [5]. In its basic form $LG$ extends Lambek’s [3] (non-unital, non-associative, non-commutative) residuated triple $\otimes, /, \backslash$ with a dual residuated triple $\oplus, \odot, \oslash$ (multiplicative sum, left and right difference). This core system can be structurally extended with dual $\otimes$ and $\oplus$ semi-associativities, and mixed semi-associativity in the sense of Grishin’s linear distributivity of the type $(A \otimes B) \otimes C \vdash A \otimes (B \otimes C)$.

References


