The Tamari order for D^3 and derivability in semi-associative Lambek-Grishin Calculus

Michael Moortgat, Utrecht Institute of Linguistics

A k-dimensional Dyck language, D^k, consists of words over a k-letter alphabet (lexicographically ordered) satisfying the following two conditions:

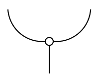
- MULTIPLICITY: each word contains the k letters with equal frequency
- PREFIX: for every prefix of a word, the number of $a_1 \geq$ the number of $a_2 \geq \ldots \geq$ the number of a_k

The familiar language of balanced brackets is D^2, with alphabet $\{a, b\}$, reading a, b as opening and closing bracket respectively. We write D^k_n for the finite D^k language with letter multiplicity n. For the place of D^k in the extended Chomsky hierarchy, the conjecture is that D^k languages are recognizable by $(k-1)$-MCFG (multiple context-free grammars, [7]). D^2 is recognized by a 1-MCFG, in other words, a simple CFG.

I report on work in progress on extending the Tamari order for D^2 to the three-dimensional case, and on modelling the Tamari order for D^3 in terms of derivability in an extended version of Lambek's Syntactic Calculus [4], going back to V.N. Grishin [2].

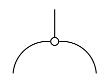
In [6], it is shown how words of D^k_n can be interpreted as the Yamanouchi words of rectangular $k \times n$ standard Young tableaux. For the three-dimensional case, Borie [1] discusses a bijection between 3-row, n-column rectangular Young tableaux and $PC(n)$, single-input single-output product coproduct prographs with n product (hence also n coproduct) nodes.

coproduct: single input, two outputs



$\Delta : V \longrightarrow V \otimes V$

product: two inputs, single output

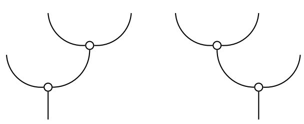


$\mu : V \otimes V \longrightarrow V$

The bijection is obtained by means of a depth-left first traversal of a PC graph with the condition that the output edge of a product node can be visited only if its two incoming edges have already been visited. Inputs of coproducts provide the entries of the top row of the corresponding tableau; left inputs of products give the entries of the middle row; right inputs of products of the bottom row. (For the Yamanouchi word over alphabet $\{a, b, c\}$ associated with a tableau, the top row entries give the positions where letter a appears, the middle row letter b, the bottom row letter c.) As an illustration, on the left of Figure 1, the Young tableau with Yamanouchi word *aabbacbcc* $\in D^3_3$ and the corresponding $PC(3)$ graph with the steps of its depth-left first traversal.

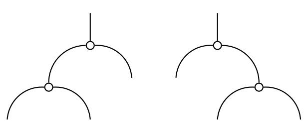
An order on the $PC(n)$ graphs, extending the two-dimensional Tamari order discussed in [9] to three dimensions can be obtained by means of the local transformations below: (a) α^Δ coproduct semi-associativity (cf left rotation in [9]), (b) α_μ product semi-associativity (dual to the above, 180° rotation), (c) α^Δ_μ mixed product-coproduct semi-associativity. The transformations (a)-(c) induce a lattice for the D^3_n order with $(abc)^n$ as minimum, and $a^n b^n c^n$ as maximum.

(a)



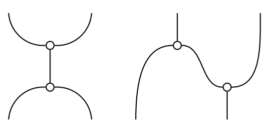
$\alpha^\Delta : (1 \otimes \Delta) \circ \Delta \longrightarrow (\Delta \otimes 1) \circ \Delta$

(b)



$\alpha_\mu : \mu \circ (\mu \otimes 1) \longrightarrow \mu \circ (1 \otimes \mu)$

(c)



$\alpha^\Delta_\mu : \Delta \circ \mu \longrightarrow (\mu \otimes 1) \circ (1 \otimes \Delta)$

As an example, the graph for the word *aabbacbcc* in Fig 1 matches the input conditions for coproduct semi-associativity (marked red) and for product semi-associativity (blue). The results of rewriting with the associated Young tableau and its Yamanouchi word are shown in the middle and on the right.

Zeilberger [9] shows that $A \leq B$ in the Tamari order for D^2 can be modelled as logical derivability $A \vdash B$ in a variant of Lambek's Syntactic Calculus with a restricted form of associativity. I investigate

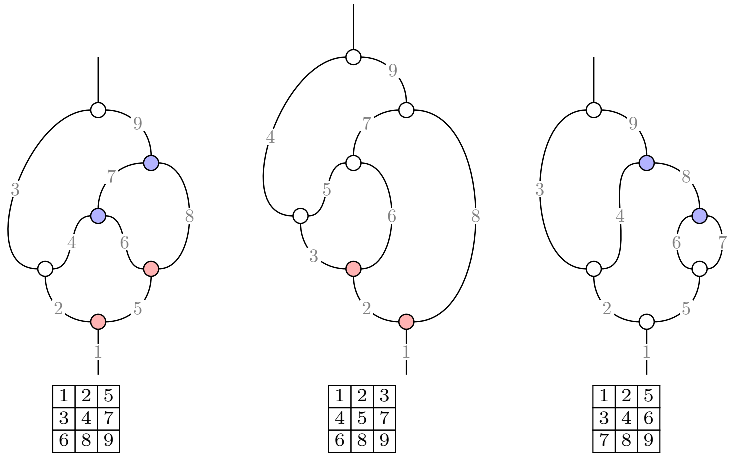


Figure 1: Graph rewriting: $aabbacbcc \xrightarrow{\alpha^{\Delta}} aaabbcbcc$ (red), $aabbacbcc \xrightarrow{\alpha_{\mu}} aabbabccc$ (blue)

whether a similar logical perspective on the D^3 order can be obtained in a suitably restricted form of the Lambek-Grishin calculus **LG** [5]. In its basic form **LG** extends Lambek's [4] (non-unital, non-associative, non-commutative) residuated triple $\otimes, /, \backslash$ with a dual residuated triple $\oplus, \oslash, \oslash$ (multiplicative sum, left and right difference). This core system can be structurally extended with dual $\otimes$ and $\oplus$ semi-associativies, and *mixed* semi-associativity in the sense of Grishin's linear distributivity of the type $(A \oslash B) \otimes C \vdash A \oslash (B \otimes C)$.

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