Statistical properties of random lambda-terms in de-Bruijn notation*

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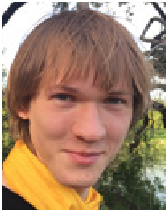
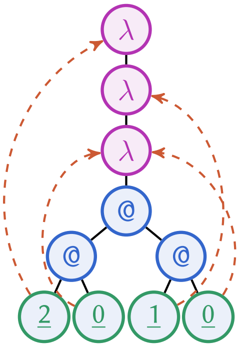


CLA-2017, Götenburg, Sweden

∗ in progress

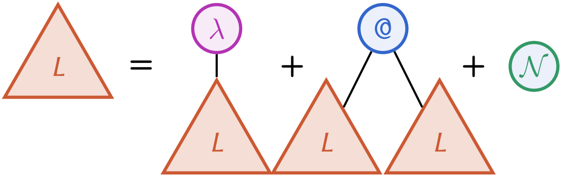
Outline

Example of lambda-term in de-Bruijn notation



A closed lambda-term $\lambda x.\lambda y.\lambda z.xz(yz)$

Grammar of plain lambda-terms



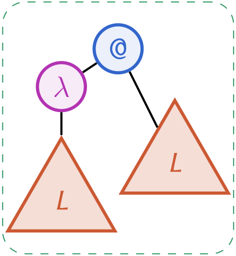
L — plain lambda-term

λ — abstraction

@ — application

$\mathcal{N}$ — variable; de Bruijn index $\in \{0, 1, 2, \ldots\}$

Redex and beta-reduction

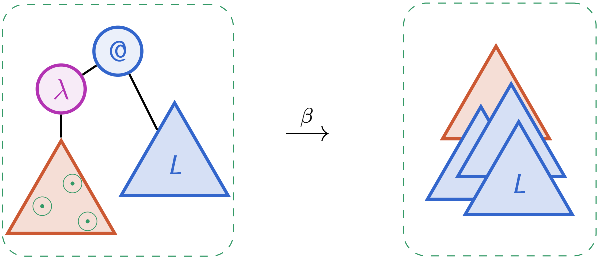


$$(\lambda n.n \times 2)7 \xrightarrow{\beta} 7 \times 2$$

$$\Omega = (\lambda x.xx)(\lambda x.xx)$$

$$\Omega \xrightarrow{\beta} \Omega$$

Redex and beta-reduction



Main question

Investigate statistical properties of random plain / closed lambda-terms in de-Bruijn notation:

- number of lambdas, variables, abstractions,. . .
- length to the leftmost outermost *redex*,
- unary height, longest lambda-run
- . . .

Statistical properties

Random generation

$\Rightarrow$ Property-based testing

Size notion of lambda-terms

- [Bodini, Gardy, Gittenberger, Jacquot '13]
  Closed lambda-terms with variable size $= 1$.
- [David, Grygiel, Kozik, Raffalli, Theyssier, Zaionc '13]
  Closed lambda-terms with variable size $= 0$.
- [Gittenberger, Gołębiewski '16]
  Natural counting of lambda-terms.

$$|0| = a, \quad |S| = b, \quad |\lambda| = d, \quad |@| = d$$

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What is "random"?

Thanks to previous talks

- *Natural size notion* of lambda-term.

  Stay tuned for the definition and comparison to other models.

- Sample lambda-terms of size n **uniformly**

  (*leitmotif* of this talk, but not the only possibility)

- Sample lambda-terms of size n and parameter value k uniformly.

  Bivariate generating function + tuning of Boltzmann sampler.

- Choose *any subset* of parameters, fix their values and sample at uniform from the desired set.

  [Bodini, Ponty '10]: Newton iteration or asymptotic approximations

  [Bodini, D. '17]: Fast exact tuning in $O\Big(\log^2(\text{size}) \cdot \#\text{vars}^7 \cdot (\#\text{vars}+\#\text{eqs})\Big)$.

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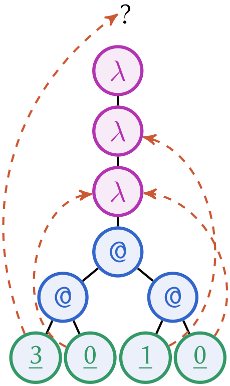
Closed lambda-terms?



The value of each index shouldn't exceed maximal unary distance to parent lambda.

m-open lambda-terms

- **Def.** A lambda-term T is *m-open* if $\lambda^m T$ is closed.
- **Observation.** $m = 0$ corresponds to closed terms
- **Def.** L_m – class of m-open lambda-terms.
- **Def.** L_∞ – class of plain lambda-terms.
- **Observation.** $L_0 \subset L_1 \subset L_2 \subset \ldots \subset L_\infty$

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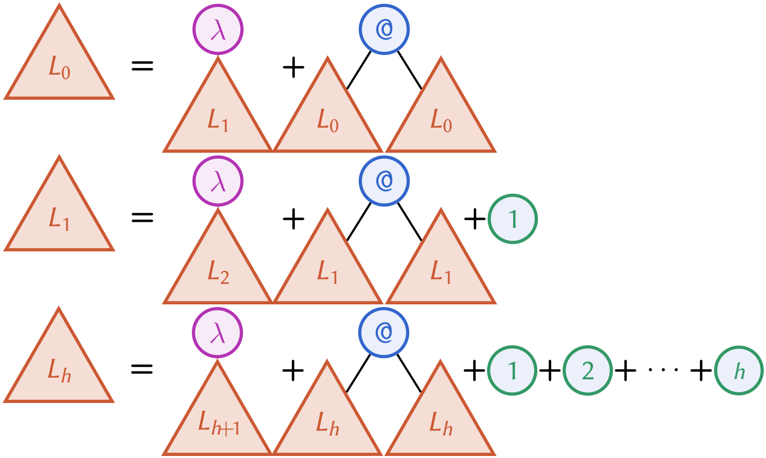
Closed terms specification



Asymptotic number of plain lambda-terms

Theorem. As $n \to \infty$, the number of plain lambda-terms of size n is asymptotically

$$\frac{b_\infty n^{-3/2}}{2\sqrt{\pi}} \left(\frac{1}{\rho}\right)^n, \quad (1-\rho)^3 = 4\rho^2 \ .$$

$$\rho \approx 0.29559$$

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Theorem. The asymptotic probability that a random plain lambda-term of size n is m-open tends to some positive constant p_m as $n \to \infty$. This distribution is *computable.*

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Open question. What is the behaviour of the sequence $(p_k)_{k=0}^{\infty}$ which is a cumulative distribution function? We only know that $p_{m+1} \geq p_m$ and $p_m \to 1$ as $m \to \infty$.

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Open question. What is the behaviour of the sequence $(p_k)_{k=0}^{\infty}$ which is a cumulative distribution function? We only know that $p_{m+1} \geq p_m$ and $p_m \to 1$ as $m \to \infty$. The reccurence can be considered either forward or backwards:

$$\begin{cases} a_{m+1} &= a_m/\rho - a_m^2 - \dfrac{1-\rho^m}{1-\rho}, \\ b_{m+1} &= b_m/\rho - 2a_m b_m, \\ p_m &= b_m/b_\infty \end{cases}$$

Outline

Zoo of different statistics

Marking techniques
Number of lambdas
Number of variables
Number of abstractions
Number of redexes
Value of de-Bruijn index
Number of head abstractions

Extremal techniques
Maximal de-Bruijn index value
Unary height of a random term
Longest lambda-run

Advanced marking
Number of free variables
Number of closed subterms
Expected search time for β-reduction
Number of variables bound to top lambda

Marking techniques

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Statistics with gaussian distribution

$\boxed{\textbf{Theorem.}}$ In random plain lambda-term of size n

- $\mathbb{E}_n(\#\ \text{lambdas}) = C_\lambda \cdot n + O(n^{1/2})$,
- $\mathbb{E}_n(\#\ \text{applications}) = C_@ \cdot n + O(n^{1/2})$,
- $\mathbb{E}_n(\#\ \text{variables}) = C_\mathcal{N} \cdot n + O(n^{1/2})$,
- $\mathbb{E}_n(\#\ \text{redexes}) = C_{\text{redex}} \cdot n + O(n^{1/2})$,

The distribution is asymptotically Gaussian, i.e.

$$\boxed{\frac{\# - \mathbb{E}\#}{\mathbb{V}\#} \xrightarrow{d} \mathcal{N}(0,1)}$$

Statistics with gaussian distribution

$\boxed{\textbf{Theorem.}}$ In random **closed** lambda-term of size n

- $\mathbb{E}_n(\#\text{ lambdas}) = C_\lambda \cdot n + O(n^{1/2})$,
- $\mathbb{E}_n(\#\text{ applications}) = C_@ \cdot n + O(n^{1/2})$,
- $\mathbb{E}_n(\#\text{ variables}) = C_\mathcal{N} \cdot n + O(n^{1/2})$,
- $\mathbb{E}_n(\#\text{ redexes}) = C_{\text{redex}} \cdot n + O(n^{1/2})$,

The constants C_λ, $C_@$, $C_\mathcal{N}$, C_{redex} are the same as in the plain case.

$$\boxed{\frac{\# - \mathbb{E}\#}{\mathbb{V}\#} \xrightarrow{d} \mathcal{N}(0,1)}$$

Head abstractions

Theorem.

- The number of head abstractions in random plain lambda-term has a limiting distribution $Geom(\rho)$, i.e.

$$\mathbb{P}(\#\text{ head abstractions } \leq m) = 1 - \rho^{m+1}$$

- The number of head abstractions in random **closed** lambda-term has cumulative distribution function

$$\mathbb{P}(\#\text{ head abstractions } \leq m) = 1 - \rho^{m+1}\frac{p_{m+1}}{p_0}$$

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Distribution of de-Bruijn index

Theorem.

- In large random plain lambda-terms, the value of de Bruijn index has a limiting distribution which is *Geom*(ρ).
- In large random **closed** lambda-terms, the value of de Bruijn index has a *computable* limiting distribution.

$$\mathbb{P}_m \sim \frac{\rho^{2m}}{(1 - 2\rho a_0) \dots (1 - 2\rho a_{m-1})}$$

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Advanced marking

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Number of free variables
Number of closed subterms
Expected search time for β-reduction
Number of variables bound to head lambda

Number of free variables

Theorem. Inside large plain lambda-terms the number of free variables has a *computable* discrete limiting distribution, in particular, the average number of free variables is a constant

$$\mathbb{E}_n \sim \frac{2}{(1-\rho)^3}$$

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"Paradox" Almost all the variables are bounded but not all the terms are closed!

Intuition. Distribution of db-index is geometric, but unary height is $O(\sqrt{n})$. A small proportion of variables makes the term open with positive probability.

Number of closed subterms

Theorem. Inside large plain (also closed) lambda-terms the number of closed subterms satisfies

$$\mathbb{E}_n \sim \Theta(n), \quad \mathbb{V}_n \sim \Theta(1)$$

Open question. What is the distribution?

Expected search time for β-reduction

Theorem. Inside large plain lambda-terms the number of steps until first redex discovery has a *computable* discrete limiting distribution, in particular, the expected time is a computable constant.

Average number of variables bound to top lambda

Theorem. Inside large plain lambda-terms choose an abstraction uniformly at random among abstractions at unary height 1.

- $\mathbb{E}$ (number of vars bound by top lambda) $\sim C$
- The same holds for lambda at fixed unary height m. The constant is not necessary the same.

Open question 1.

- Limiting discrete distribution?
- The same holds for closed lambda-terms?

Open question 2.* Number of binding lambdas?

Extremal techniques

- Maximal de-Bruijn index value
- Unary height of a random term
- Longest lambda-run

Extremal statistics

Theorem. In random plain lambda-term of size n

- $\mathbb{E}_n(\text{longest lambda-run}) \sim \dfrac{\log n}{\log(1/\rho)} + O(\log\log n)$,
- $\mathbb{E}_n(\text{maximal de Bruijn index}) \sim \dfrac{\log n}{\log(1/\rho)} + O(\log\log n)$,
- $\mathbb{E}_n(\text{unary height}) \sim \Theta(\sqrt{n})$.

Conjecture. The same is true for closed lambda-terms.

Outline

Open problems

Supposedly hard

- Combinatorics of lambda-term after beta-reduction procedure
- Closed BCI, BCK, λ-I terms.
  Each lambda binds $(\cdot)$ variables
  - ≤ 1 BCI
  - $= 1$ λ-I
  - ≥ 1 BCK

  Riccati PDE, multivariate saddle-point, etc.

  [Lescanne '17] SwissCheese: keeping a large vector of information to track the number of variables on every level
- Number of binding lambdas
- Phase transitions with respect to *abcd* size notion

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That's all!

Thank you for your attention!