Shifting the thresold of phase transition in 2-SAT and random graphs

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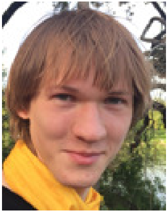

Acknowledgements: Élie de Panafieu, Fedor Petrov, `ipython+sympy+cpp`

May 19, 2017

Outline

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Phase transition in Erdős–Rényi random graphs

n vertices, m edges,

$$m = \frac{1}{2}n(1 + \mu n^{-1/3})$$

1. "gas" $\mu \to -\infty$: planar graph, trees and unicycles, max component size $O(\log n)$.
2. "liquid" $|\mu| = O(1)$: complex components appear, max component size $O(n^{2/3})$.
3. "crystal" $\mu \to +\infty$: non-planar, complex compontnes, max component size linear $O(n)$.

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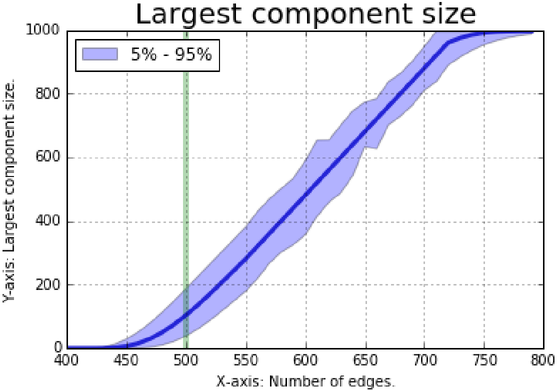
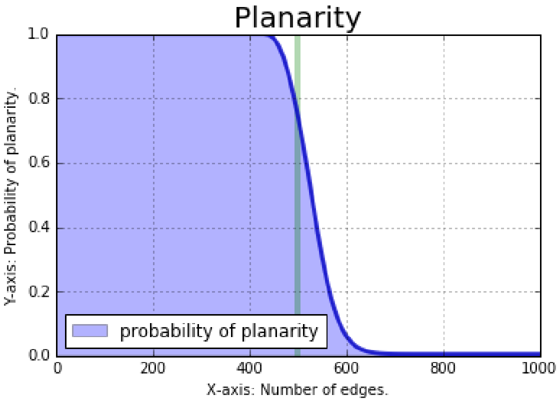
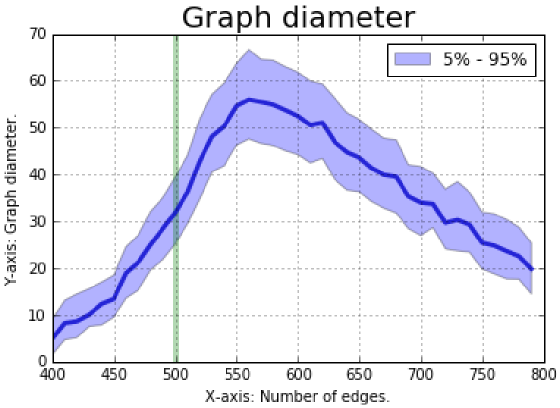
Phase transition :: largest component, $n = 1000$



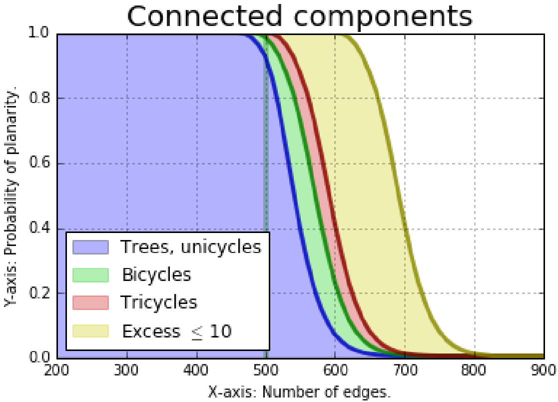
Phase transition :: planarity, $n = 1000$



Phase transition :: diameter, $n = 1000$



Phase transition :: connected components, $n = 1000$



2SAT Transition

1. [Bollobás, Borgs, Chayes, Kim, and Wilson '99]
   2SAT Transition
2. [Coppersmith, Gamarnik, Hajaghayi, Sorkin '03]
   MAX 2-SAT Transition
3. [Cooper, Freize, Sorkin '07]
   2SAT with degree sequence constraints

Shifting the phase transition

$$m = \frac{1}{2} n(1 + \mu n^{-1/3}) \Rightarrow m = \alpha n(1 + \mu n^{-1/3})$$

1. Achlioptas percolation process ($\alpha = 0.535?$)
2. Degree sequence models (less detailed information)
3. Degree set constraint :: current talk

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Example of graph with degree constraints

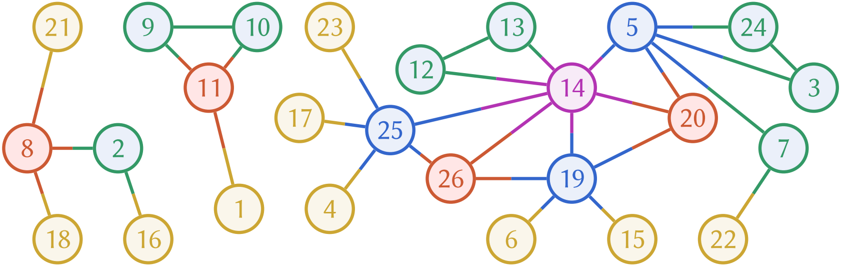

Figure: Random labeled graph from $\mathcal{G}_{26,30,\Omega}$ with the set of degree constraints $\Omega = \{1, 2, 3, 5, 7\}$.

Constant of phase transition

Ω — the set of degree constraints

1. Random graphs

$$m = \frac{1}{2} n(1 + \mu n^{-1/3}) \overset{?}{\Rightarrow} m = \alpha n(1 + \mu n^{-1/3})$$

2. Random 2-CNF

$$m = 1 \cdot n(1 + \mu n^{-1/3}) \overset{?}{\Rightarrow} m = 2\alpha n(1 + \mu n^{-1/3})$$

3. How to compute α depending on Ω?

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Exponential generating function

1. Set of degree constraints. $\Omega = \{1, 2, 3, 5, 7\}$. Can be infinite.
2. Exponential generating function connected to Ω

$$\omega(z) = \sum_{d \in \Omega} \frac{z^d}{d!} = \frac{z^1}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!}.$$

3. Definition of the point $\alpha(\Omega)$:

$$\begin{cases} \widehat{z}\dfrac{\omega''(\widehat{z})}{\omega'(\widehat{z})} = 1, \\ \widehat{z}\dfrac{\omega'(\widehat{z})}{\omega(\widehat{z})} = 2\alpha \end{cases} \tag{1}$$

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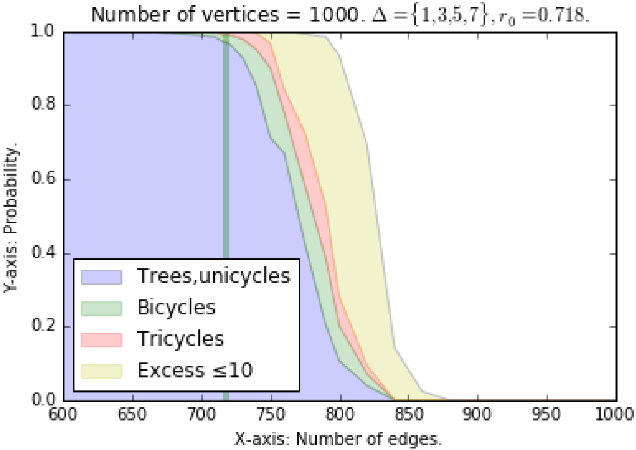
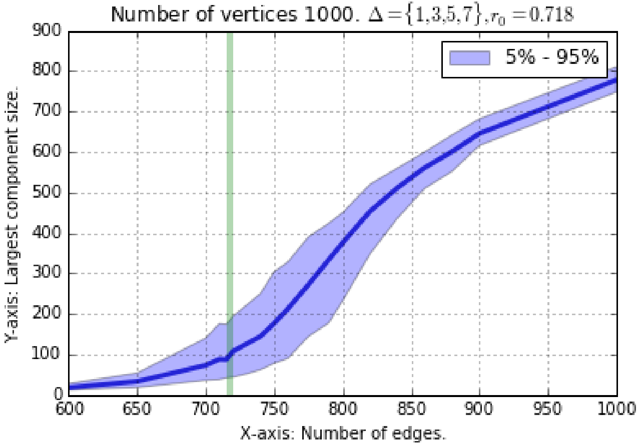
Experimental results

(1/3)



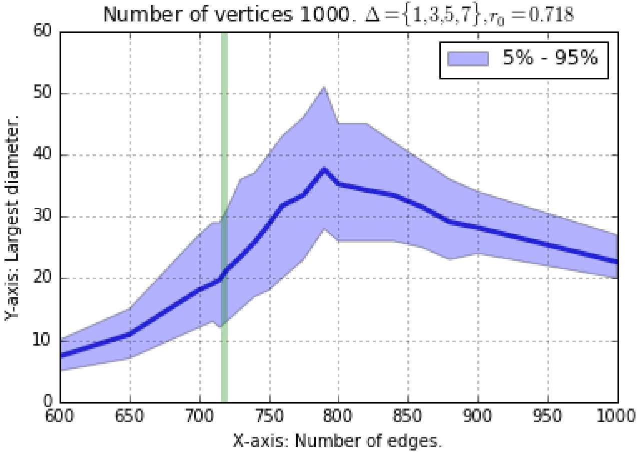
Experimental results

(2/3)



Experimental results

(3/3)



Ipython session :: let's compute the threshold point!

Outline

Precise statement of the theorem

Theorem Let $F_{n,m,\Omega}$ be random 2-CNF with Ω-degree constraints.
n – number of variables
m – number of clauses

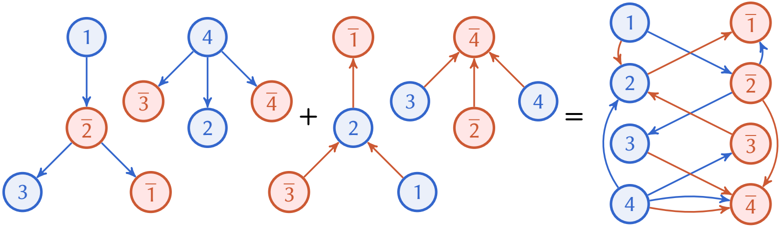
$$m = \alpha n(1 + \mu n^{-1/3})$$

1. $\mathbb{P}(F_{n,m,\Omega} \text{ is SAT}) \geq 1 - O(|\mu|^{-3})$ as $\mu \to -\infty$,
2. $\mathbb{P}(F_{n,m,\Omega} \text{ is SAT}) \geq \Theta(1)$ as $|\mu| = O(1)$,
3. $\mathbb{P}(F_{n,m,\Omega} \text{ is SAT}) \geq \exp(-\Theta(\mu^3))$ as $\mu \to +\infty$.

2-CNF formula and digraph model

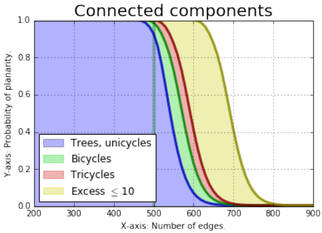
Digraph representation and sum-representation of a 2-SAT formula

$$(\overline{x}_1 \vee \overline{x}_2)(x_2 \vee x_3)(x_2 \vee \overline{x}_1)(\overline{x}_4 \vee \overline{x}_3)(\overline{x}_4 \vee x_2)(\overline{x}_4 \vee \overline{x}_4)$$



Tools from random graphs

n – number of vertices
m – number of edges



Framework: $m = \alpha n$, linear dependence.

1. $m = (1-\varepsilon)\alpha n \quad \longleftarrow$ only trees and unicycles
2. $m = \alpha n \quad \longleftarrow$ complex components with positive probability
3. $m = (1+\varepsilon)\alpha n \quad \longleftarrow$ probability of fixed excess is exponentially small

Structural theorem for random graphs

Theorem (Regime: $m = \alpha n(1 + \mu n^{-1/3})$)

1. *if* $\mu \to -\infty$, $|\mu| = O(n^{1/12})$, *then*

   $$\mathbb{P}(G_{n,m,\Omega} \text{ has only trees and unicycles}) = 1 - \Theta(|\mu|^{-3}) \;;$$

2. *if* $|\mu| = O(1)$, *i.e.* μ *is fixed, then*

   $$\mathbb{P}(G_{n,m,\Omega} \text{ has only trees and unicycles}) \to \text{constant} \in (0,1) \;,$$

   $$\mathbb{P}(G_{n,m,\Omega} \text{ has a complex part with total excess } q) \to \text{constant} \in (0,1) \;,$$

3. *if* $\mu \to +\infty$, $|\mu| = O(n^{1/12})$, *then*

   $$\mathbb{P}(G_{n,m,\Omega} \text{ has only trees and unicycles}) = \Theta(e^{-\mu^3/6}\mu^{-3/4}) \;,$$

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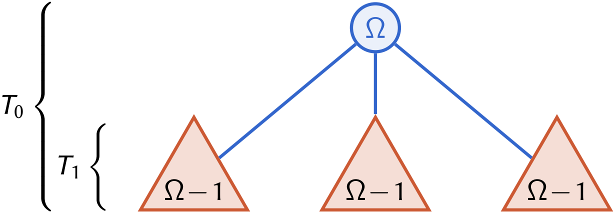
Outline

Trees with degree constraints

Rooted case



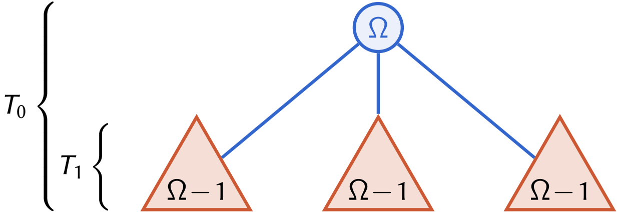
$$\Omega - k \stackrel{def}{=} \{d \colon d + k \in \Omega\}$$

Example:

$$\Omega = \{0, 1, 3, 6\}$$
$$\Omega - 1 = \{0, 2, 5\}.$$

Trees with degree constraints

Rooted case



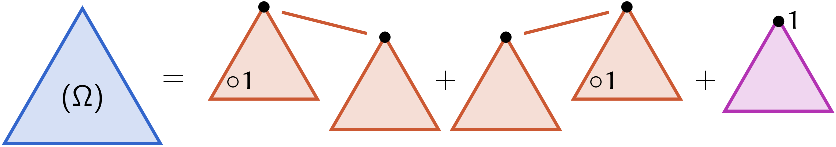
$$\begin{cases} \omega(z) &= \sum_{d\in\Omega} \frac{z^d}{d!} = \frac{z^{d_1}}{d_1!} + \frac{z^{d_2}}{d_2!} + \dots, \\ \omega'(z) &= \sum_{d\in\Omega} \frac{z^{d-1}}{(d-1)!} = \sum_{d\in\Omega-1} \frac{z^d}{d!} \end{cases},$$

$$\begin{cases} T_0(z) = z\omega(T_1(z)), \\ T_1(z) = z\omega'(T_1(z)), \\ T_2(z) = z\omega''(T_1(z)). \end{cases}$$

Trees with degree constraints

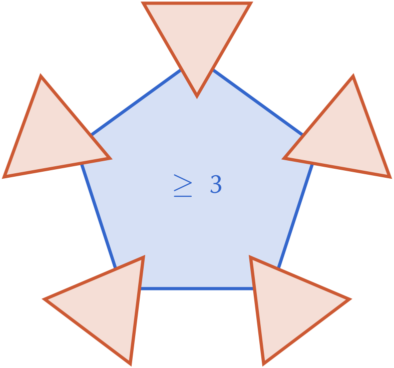
Unrooted case

A variant of dissymmetry theorem:



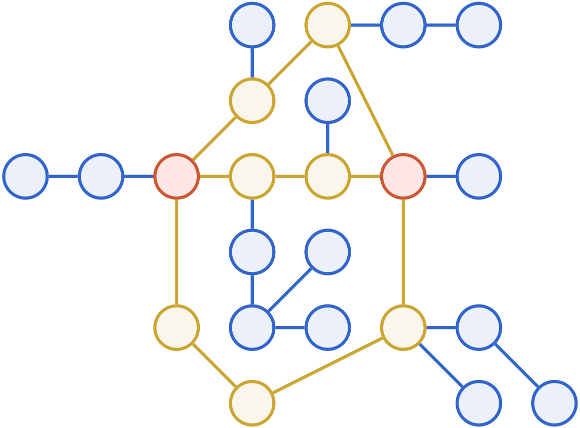
$$\underbrace{T_0(z)}_{\text{root deg. } 0} = \frac{T_1(z)^2}{\underbrace{2}_{\text{root deg. } 1}} + \underbrace{U(z)}_{\text{unrooted}} \quad \Leftrightarrow \quad U(z) = T_0(z) - \frac{T_1(z)^2}{2}$$

Unicycles with degree constraints



$$\underset{\substack{\uparrow \\ \text{unicycles}}}{V(z)} = \frac{1}{2}\left[\underset{\substack{\uparrow \\ \text{cycle}}}{\log} \frac{1}{1 - T_2(z)} - T_2(z) - \frac{T_2(z)^2}{2}\right]$$

2-core (the core) and 3-core (the kernel)



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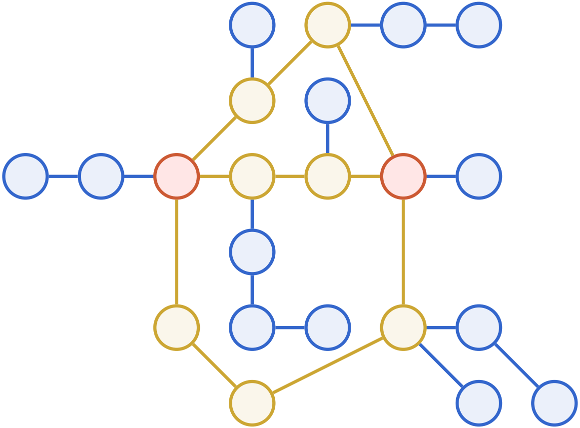

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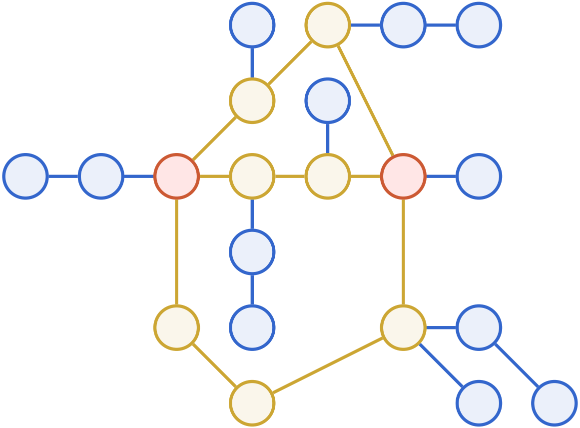

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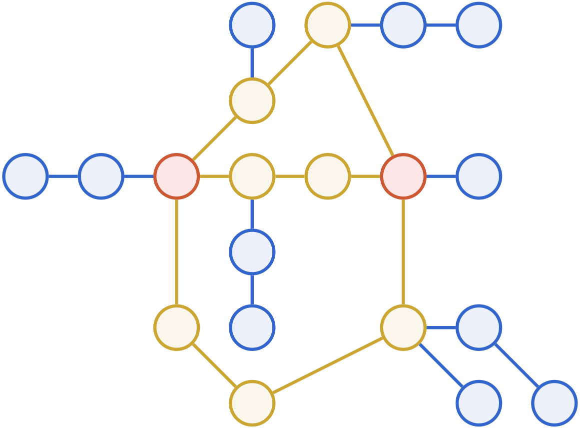

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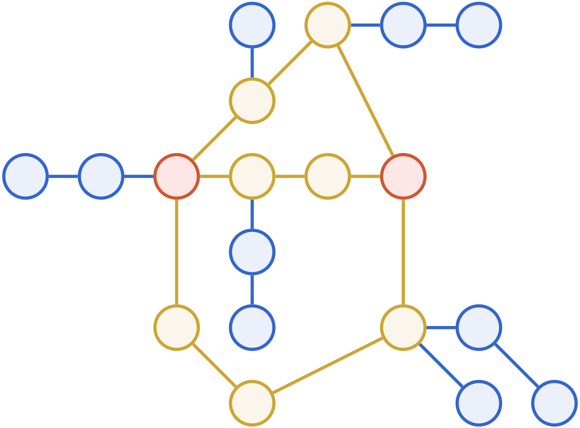

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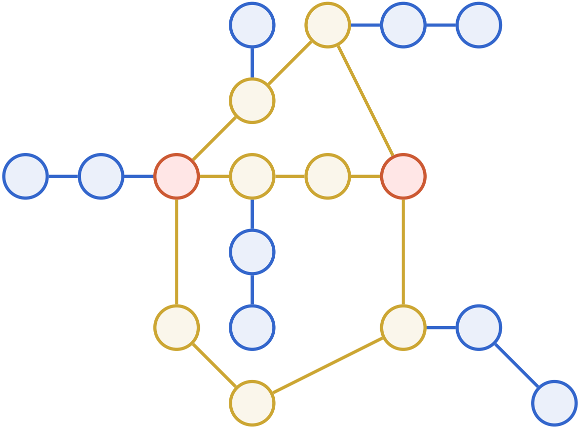

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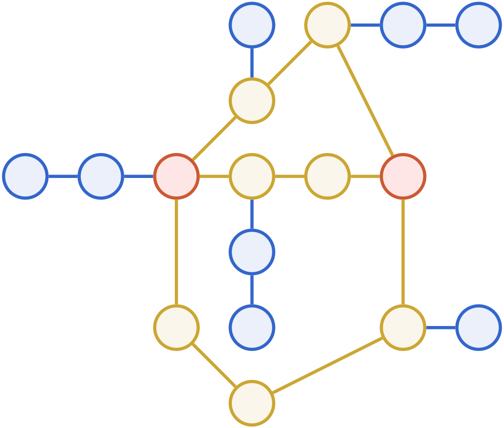

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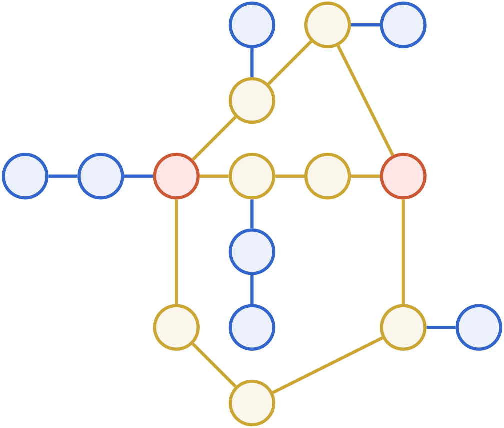

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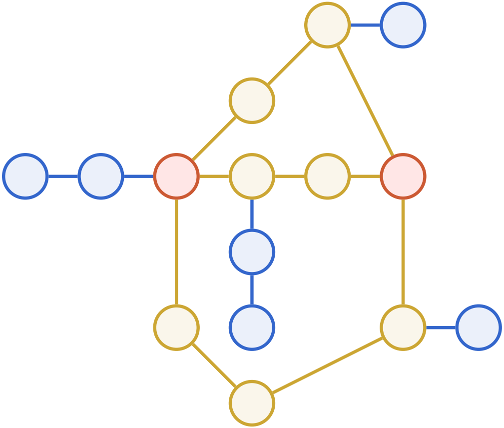

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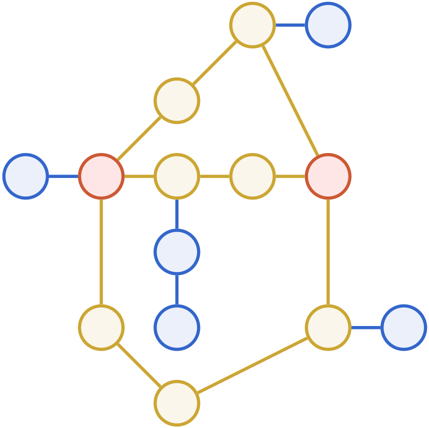

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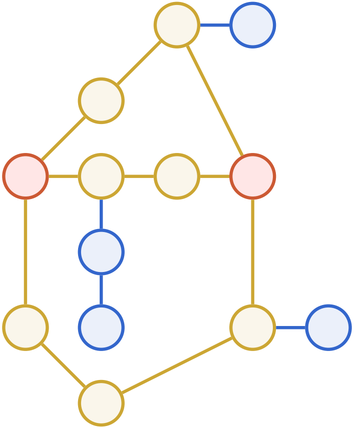

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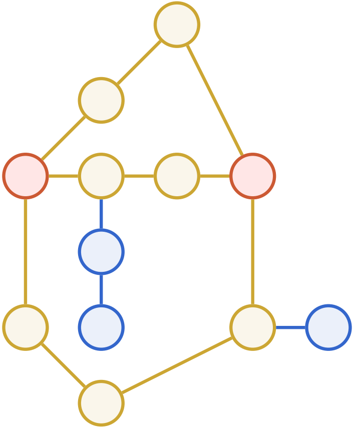

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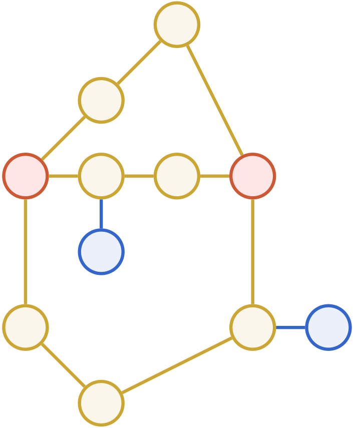

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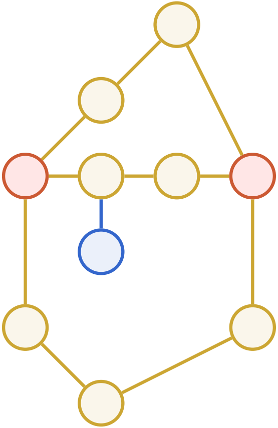

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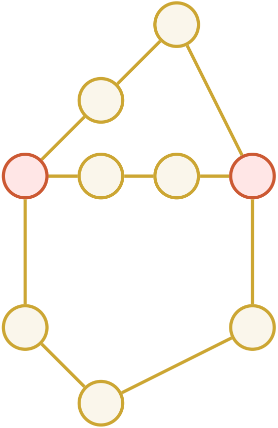

2-core of a graph

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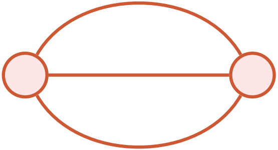
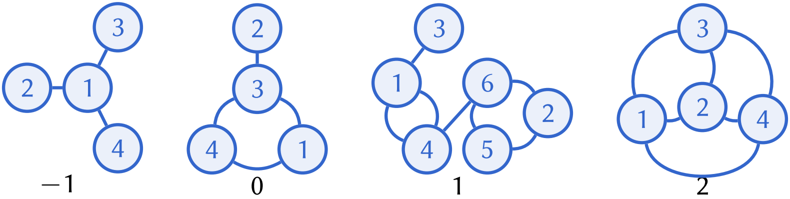


3-core of a graph

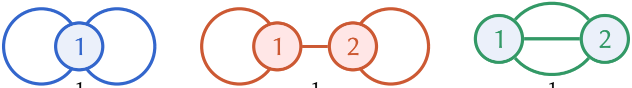
Notion of excess



$$\text{Excess} \stackrel{def}{=} \#\text{ edges - }\#\text{ vertices}$$

Kernel of a graph

Example: graphs with excess 1



$\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{6}$

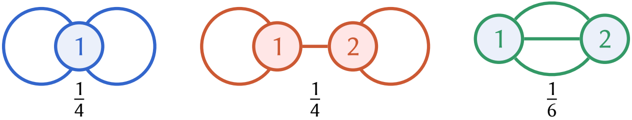
All possible 3-core multigraphs and their compensation factors.
EGF for all connected bicyclic graphs ($\Omega = \mathbb{Z}_{\geq 0}$):

$$W(z) = \frac{1}{4}\frac{T(z)^5}{(1-T(z))^2} + \frac{1}{4}\frac{T(z)^6}{(1-T(z))^3} + \frac{1}{6}\underbrace{\frac{T(z)^2[3T(z)^2 - 2T^3(z)]}{(1-T(z))^3}}_{\text{inclusion-exclusion}}$$

$$W(z) \sim \frac{5}{24} \cdot \frac{1}{(1-T(z))^3} \text{ near } z = e^{-1}$$

Kernel of a graph

Example: graphs with excess 1



All possible 3-core multigraphs and their compensation factors.
EGF for all connected bicyclic graphs (arbitrary Ω):

$$W_\Omega(z) = \frac{1}{4}\frac{T_4(z)T_2(z)^4}{(1-T_2(z))^2} + \frac{1}{4}\frac{T_3(z)^2T_2(z)^4}{(1-T_2(z))^3} + \frac{1}{6}\underbrace{\frac{T_3(z)^2[3T_2(z)^2 - 2T_2(z)^3]}{(1-T_2(z))^3}}_{\text{inclusion-exclusion}}$$

$$W_\Omega(z) \sim (???) \cdot \frac{T_3(z)^2(???)}{(1-T_2(z))^3}$$

Role of cubic graphs

EGF for all (not necessary connected) complex multigraphs with excess r,

$$W_{\Omega,r}(z) \sim e_{r0} \underbrace{\frac{T_3(z)^{2r}}{(1 - T_2(z))^{3r}}}_{\uparrow \text{ comes from cubic graphs}}, \quad e_{r0} = \underbrace{\frac{(6r)!}{2^{5r} 3^{2r} (3r)! (2r)!}}_{\uparrow \text{ can be shown combinatorially}}.$$

Local summary

1. EGF for unrooted trees with degree constraints
2. EGF for unicycles with degree constraints
3. EGF for graphs of fixed *excess* (main asymptotics)

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Desired probability

Subcritical phase

$$\mathbb{P}\,(\text{graph } g \in \mathcal{G}(n, m, \Omega) \text{ consists only of trees and unicycles})$$

$$= \frac{\#\text{ graphs from } \mathcal{G}(n, m, \Omega) \text{ whose components are trees and unicycles}}{\#\text{ graphs from } \mathcal{G}(n, m, \Omega)}$$

Number of graphs with degree constraints

1. $\Omega = \mathbb{Z}_{\geq 0}$. Stirling approximation:

$$\frac{n!}{(n-m)!\binom{\binom{n}{2}}{m}} \sim \sqrt{4\pi n\alpha} \cdot \frac{2^m n^n m^m}{n^{2m}(n-m)^{n-m}} \times \exp\Big(-n + \underbrace{\frac{m}{n} + \frac{m^2}{n^2}}_{3/4} \Big)$$

2. Arbitrary Ω ([de Panafieu, Ramos '16])

$$\frac{n!}{(n-m)!|\mathcal{G}_{n,m,\Omega}|} \sim \frac{\sqrt{4\pi n\alpha}}{p} \cdot \frac{2^m n^n m^m}{n^{2m}(n-m)^{n-m}} \times \exp\Big(-n\log\omega(\widehat{z}) + 2m\log\widehat{z} + \underbrace{\frac{1}{2}\phi_0(\widehat{z}) + \frac{1}{4}\phi_0^2(\widehat{z})}_{3/4} \Big)$$

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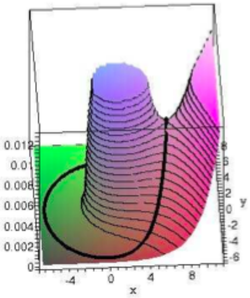
Example: contour integral for subcritical phase

$$\frac{n!}{|\mathcal{G}_{n,m,\Omega}|} \frac{1}{2\pi i} \oint \underbrace{\frac{U(z)^{n-m}}{(n-m)!}}_{\uparrow \text{ trees}} \underbrace{e^{V(z)}}_{\uparrow \text{ unicycles}} \frac{dz}{z^{n+1}} = 1 - O(\mu^{-3})$$

near the critical point $m = \alpha n$:

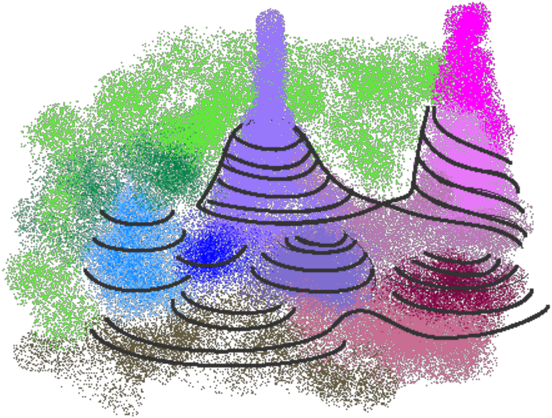
$$\begin{cases} 2\alpha & = \phi_0(\widehat{z}) \stackrel{def}{=} \widehat{z} \dfrac{\omega'(\widehat{z})}{\omega(\widehat{z})} \quad , \\ 1 & = \phi_1(\widehat{z}) \stackrel{def}{=} \widehat{z} \dfrac{\omega''(\widehat{z})}{\omega'(\widehat{z})} \quad . \end{cases}$$

Contour integral: Erdős–Rényi case



Picture from Flajolet's book

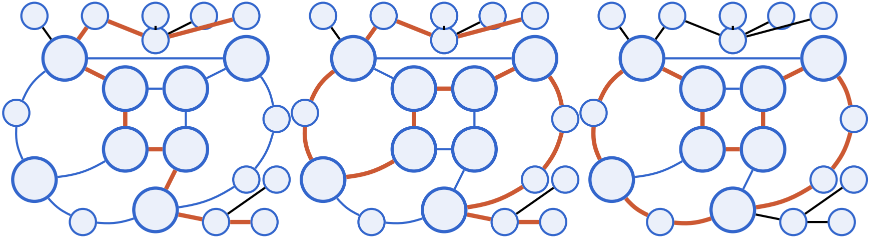
Contour integral: graphs with degree constraints



Not a single mountain but dangerous expedition!

Outline

Diameter, circumference and longest path of complex component



All of order $\Theta(n^{1/3})$

Planarity

Let $p(\mu)$ be the probability that $G_{n,m,\Omega}$ is planar.

1. $p(\mu) = 1 - \Theta(|\mu|^{-3})$, as $\mu \to -\infty$;
2. $p(\mu) \to$ constant $\in (0, 1)$, as $|\mu| = O(1)$, and $p(\mu)$ is computable;
3. $p(\mu) \to 0$, as $\mu \to +\infty$.

Summary

1. Analytic description of phase transition in model with degree constraints
2. $\frac{1}{2}$ proof of 2-SAT phase transition
3. Study of distribution of parameters.

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1. The case $1 \notin \Omega$.
2. Upper bound for 2-SAT.
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That's all!

Thank you for your attention.
Good flight back home.