Enumerating Lambda Terms By Weighted Length of Their de Bruijn Representation

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joint work with Olivier Bodini and Zbigniew Gołębiewski

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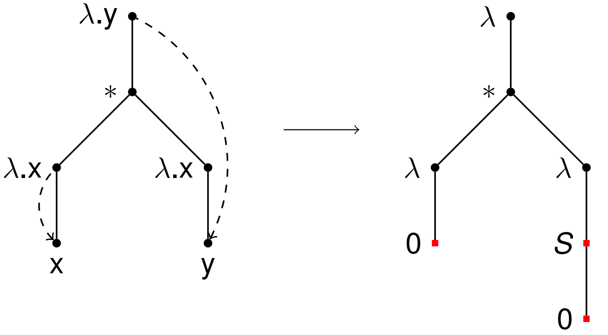
Definition of lambda terms

$$T ::= x \mid \lambda x.T \mid T * T \quad \to \quad T ::= S^n 0 \mid \lambda T \mid T * T$$

$\lambda x.T$: abstraction, unary node $\qquad$ $(T * T)$: application, binary node

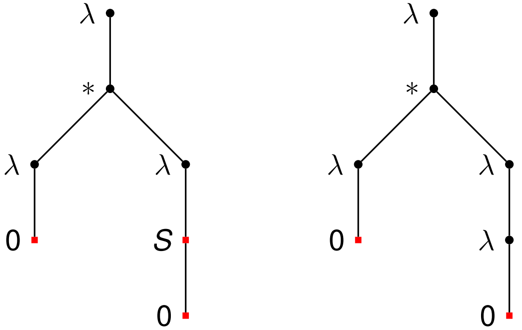
$$\lambda y.((\lambda x.x) * (\lambda x.y)) \to \lambda(\lambda 1 * \lambda 2) \to \lambda((\lambda 0) * (\lambda(S0)))$$



Definition of lambda terms

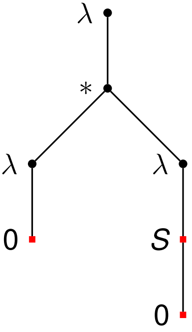
$$\lambda y.(\lambda x.x * \lambda x.y) \quad \not\equiv \quad \lambda y.(\lambda x.x * \lambda x.\lambda z.z)$$

$$\lambda((\lambda 0) * (\lambda(S0))) \quad \not\equiv \quad \lambda((\lambda 0) * (\lambda(\lambda 0)))$$

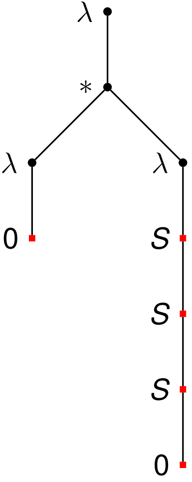


m-open lambda terms

closed lambda term (0-open)
$\lambda((\lambda 0) * (\lambda(S0)))$



2-open lambda term
$\lambda((\lambda 0) * (\lambda(SSS0)))$



General notion of size

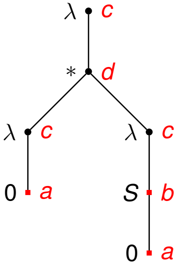
$$
\begin{aligned}
|0| &= a \\
|S| &= b \\
|\lambda M| &= |M| + c \\
|MN| &= |M| + |N| + d.
\end{aligned}
$$

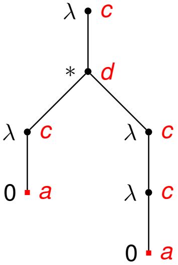
Assumptions

1. a, b, c, d *are nonnegative integers,*
2. $a + d \geq 1$,
3. $b, c \geq 1$,
4. $\gcd(b, c, a + d) = 1$.

General notion of size

$\lambda((\lambda 0) * (\lambda(S0))) \quad \neq \quad \lambda((\lambda 0) * (\lambda(\lambda 0)))$





size: $2a + b + 3c + d$

size: $2a + 4c + d$

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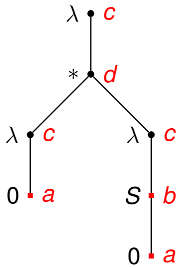

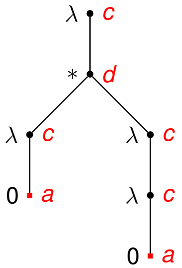

size: $2a + b + 3c + d$ size: $2a + 4c + d$

- natural counting (Bendkowski, Grygiel, Lescanne, Zaionc 2015): $a = b = c = d = 1$
- less natural counting (Bendkowski, Grygiel, Lescanne, Zaionc 2015): $a = 0,\ b = c = 1,\ d = 2$
- binary lambda calculus (Tromp 2006): $b = 1,\ a = c = d = 2$

Combinatorial specifications and generating functions

Let $(\mathcal{A}, |\cdot|_\mathcal{A})$, $(\mathcal{B}, |\cdot|_\mathcal{B})$ be comb. structures with generating functions

$$A(z) = \sum_{n\geq 0} a_n z^n = \sum_{x\in\mathcal{A}} z^{|x|_\mathcal{A}} \text{ and}$$

$$B(z) = \sum_{n\geq 0} b_n z^n = \sum_{x\in\mathcal{B}} z^{|x|_\mathcal{B}}.$$

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- If $\mathcal{A} \cap \mathcal{B} = \emptyset$ and $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ with

$$|x|_{\mathcal{C}} := \begin{cases} |x|_{\mathcal{A}} & \text{if } x \in \mathcal{A}, \\ |x|_{\mathcal{B}} & \text{if } x \in \mathcal{B}, \end{cases}$$

  then $c_n = a_n + b_n$ and $C(z) = A(z) + B(z)$.

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- If $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ and $|(a,b)|_{\mathcal{C}} = |a|_{\mathcal{A}} + |b|_{\mathcal{B}}$ then $C(z) = A(z) \cdot B(z)$.

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  then $c_n = a_n + b_n$ and $C(z) = A(z) + B(z)$.
- If $\mathcal{C} = \mathcal{A}\times\mathcal{B}$ and $|(a,b)|_\mathcal{C} = |a|_\mathcal{A} + |b|_\mathcal{B}$ then $C(z) = A(z)\cdot B(z)$.
- If $\mathcal{C} = \textsc{Seq}(A)$ and $|(a_1,\ldots,a_k)|_\mathcal{C} = \sum_{i=1}^k |a_i|_\mathcal{A}$ then $C(z) = \frac{1}{1-A(z)}$.

Combinatorial specification and lambda terms

$$\mathcal{L} = \textsc{Seq}(\mathcal{S}) \times \mathcal{Z} \cup \mathcal{U} \times \mathcal{L} \cup \mathcal{A} \times \mathcal{L}^2$$

- $\mathcal{L}$ – the class of lambda terms,
- $\mathcal{Z}$ – the class of zeros,
- $\mathcal{S}$ – the class of successors,
- $\mathcal{U}$ – the class of abstractions,
- $\mathcal{A}$ – the class of applications.

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Thus

$$L(z) = z^a \sum_{j=0}^{\infty} z^{bj} + z^c L(z) + z^d L(z)^2,$$

$[z^n]L(z)$=number of lambda terms of size n.

m-open terms and functional equations

Let

$$\mathcal{L}_m = \textsc{Seq}_{\leq m-1}(\mathcal{S}) \times \mathcal{Z} \;\cup\; \mathcal{U} \times \mathcal{L}_{m+1} \;\cup\; \mathcal{A} \times \mathcal{L}_m^2.$$

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- $L_{m,n}$ – the number of m-open lambda terms of size n,
- $L_m(z) = \sum_{n\geq 0} L_{m,n} z^n \quad ([z^n]\, L_m(z) = L_{m,n})$

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$$L_m(z) = z^a \sum_{j=0}^{m-1} z^{bj} + z^c L_{m+1}(z) + z^d L_m(z)^2$$

- $L_0(z)$ is the gen. fun. of the set $\mathcal{L}_0$ of closed lambda terms,
- $L_\infty(z)$ is the gen. fun. of the set $\mathcal{L}_\infty = \mathcal{L}$ of all lambda terms.

$L_\infty(z)$ – all terms

Solving

$$L_\infty(z) = z^a \sum_{j=0}^{\infty} z^{bj} + z^c L_\infty(z) + z^d L_\infty(z)^2.$$

we get

$$L_\infty(z) = \frac{1 - z^c - \sqrt{(1 - z^c)^2 - \frac{4z^{a+d}}{1-z^b}}}{2z^d},$$

which defines an analytic function in a neighbourhood of $z = 0$.

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Theorem (Flajolet, Odlyzko 1990)

If $\alpha \in \mathbb{R} \setminus \mathbb{N}$ and $f(z) \sim (1 - \frac{z}{\rho})^\alpha$ as $z \to \rho$ within a Δ-domain, then

$$[z^n]f(z) \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)} \rho^{-n}, \text{ as } n \to \infty.$$

$L_\infty(z)$ – all terms

Proposition

Let $\rho = \mathrm{RootOf}\left\{(1-z^b)(1-z^c)^2 - 4z^{a+d}\right\}$. *Then*

$$L_\infty(z) = a_\infty - b_\infty\sqrt{1-\frac{z}{\rho}} + \mathrm{O}\left(\left|1-\frac{z}{\rho}\right|\right),$$

for some constants $a_\infty > 0, b_\infty > 0$ *that depend on* a, b, c, d.

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Corollary

The coefficients of $L_\infty(z)$ *satisfy*

$$L_{\infty,n} \sim \frac{b_\infty}{2\sqrt{\pi}}\rho^{-n}n^{-3/2},\ \text{as } n \to \infty.$$

Theorem (G., Gołębiewski 2016)

Let $\rho = \mathrm{RootOf}\left\{(1-z^b)(1-z^c)^2 - 4z^{a+d}\right\}$. *Then there exist positive constants* $\underline{C}$ *and* $\overline{C}$ *(depending on* a, b, c, d *and* m*) such that the number of* m*-open lambda terms of size* n *satisfies*

$$\liminf_{n\to\infty} \frac{L_{m,n}}{\underline{C} n^{-\frac{3}{2}} \rho^{-n}} \geq 1 \quad and \quad \limsup_{n\to\infty} \frac{L_{m,n}}{\overline{C} n^{-\frac{3}{2}} \rho^{-n}} \leq 1,$$

Remark

In case of given a, b, c, d *and* m *we can compute numerically such constants* $\underline{C}$ *and* $\overline{C}$*.*

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Remark

In case of given a, b, c, d *and* m *we can compute numerically such constants* $\underline{C}$ *and* $\overline{C}$.
For instance, for natural counting we have

$$\underline{C}^{(nat)} \approx 0.07790995266\ldots,$$
$$\overline{C}^{(nat)} \approx 0.07790998229\ldots.$$

Key idea: Replacing $L_m(z)$

We have

$$L_m(z) = z^a \sum_{j=0}^{m-1} z^{bj} + z^c L_{m+1}(z) + z^d L_m(z)^2.$$

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$\mathcal{L}_m^{(h)}$ – lambda terms in $\mathcal{L}_m$ where the length of each string of successors is bounded by h

$$L_m^{(h)}(z) = \begin{cases} z^a \sum_{j=0}^{m-1} z^{bj} + z^c L_{m+1}^{(h)}(z) + z^d L_m^{(h)}(z)^2 & \text{if } m < h, \\ z^a \sum_{j=0}^{h-1} z^{bj} + z^c L_h^{(h)}(z) + z^d L_h^{(h)}(z)^2 & \text{if } m \geq h, \end{cases}$$

because for $m \geq h$ we have $L_m^{(h)}(z) = L_h^{(h)}(z)$.
$\rightsquigarrow$ upper and lower bounds.

Theorem

Let $\rho = \mathrm{RootOf}\left\{(1-z^b)(1-z^c)^2 - 4z^{a+d}\right\}$. *Then there exists a positive constant* C *(depending on* a, b, c, d *and* m*) such that the number of* m*-open lambda terms of size* n *satisfies*

$$L_{m,n} \sim C n^{-\frac{3}{2}} \rho^{-n}$$

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$$\lim_{m\to\infty} L_m(z) = L_\infty(z) = a_\infty - b_\infty\sqrt{1-\frac{z}{\rho}}.$$

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Replace $L_m(z)$ by $L_\infty(z)$ and trace back:

$$a_m := a_\infty, \qquad b_m := b_\infty; \qquad\qquad L_{m,m}(z) := L_\infty(z).$$

$$L_{m,m}(z) = L_\infty(z) = a_\infty - b_\infty\sqrt{1 - \frac{z}{\rho}},$$

$$L_{i,m}(z) = z^a \sum_{j=0}^{i-1} z^{jb} + z^c L_{i+1,m}(z) + z^d L_{i,m}(z)^2.$$

Then

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Lemma

The sequences $(a_{0,m})_{m\geq 0}$ *and* $(b_{0,m})_{m\geq 0}$ *are convergent.*

Proof.

We know that $\lim_{m\to\infty} L_{0,m}(z) = L_0(z)$, uniformly in $[0,\rho]$ and that $L_{0,m}(z)$ is decreasing. Thus

$$a_{0,m} = L_{0,m}(\rho) \longrightarrow L_0(\rho) =: a_0, \text{ as } m \to \infty.$$

b_0, m is increasing and bounded by b_∞, thus converges to b_0. □

The theorem follows now from the uniform convergence of $L_{0,m}(z)$ and the local shape of these functions.

Lambda terms containing q abstractions

$\mathcal{L}_{m,q}$ ... class of m-open lambda terms with exactly q abstractions,
$L_{m,q,n}$ ... number of those terms being of size n,
$L_{m,q}(z) = \sum_{n\geq 0} L_{m,q,n} z^n$.

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Then

$$L_{m,0}(z) = z^a \sum_{j=0}^{m-1} z^{bj} + z^d L_{m,0}(z)^2,$$

and thus

$$L_{m,0}(z) = \frac{1 - \sqrt{1 - 4z^{a+d} \sum_{j=0}^{m-1} z^{bj}}}{2z^d}.$$

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For general q:

$$L_{m,q}(z) = z^c L_{m+1,q-1}(z) + z^d \sum_{\ell=0}^{q} L_{m,\ell}(z) L_{m,q-\ell}(z),$$

thus

$$L_{m,q}(z) = \frac{\left(z^c L_{m+1,q-1}(z) + z^d \sum_{\ell=1}^{q-1} L_{m,\ell}(z) L_{m,q-\ell}(z)\right)}{\sqrt{1 - 4z^{a+d} \sum_{j=0}^{m-1} z^{bj}}}.$$

Lambda terms containing q abstractions

Lemma

Let $\delta_m(z) = \sqrt{1 - 4z^{a+d}\sum_{j=0}^{m-1} z^{bj}}$. *Then, for all* $m, q \geq 0$, *there exists a rational function* $R_{m,q}(z)$ *such that*

$$L_{m,q}(z) = -\frac{z^{cq}\delta_{m+q}(z)}{2z^d \prod_{i=0}^{q-1} \delta_{m+i}(z)} + R_{m,q}(z).$$

Moreover, the denominator of $R_{m,q}(z)$ *is of the form* $\prod_{i=0}^{q-1} \delta_{m+i}(z)^{\alpha_i}$ *where the exponents* $\alpha_0, \ldots, \alpha_{q-1}$ *are positive integers.*

Proof.

Induction on q. □

Lambda terms containing q abstractions

Corollary

Let $\xi_m = \mathrm{RootOf}\{1 - 4z^{a+d} \sum_{j=0}^{m-1} z^{bj}\}$ (dominant singularity of $\delta_m(z)$). Then $\xi_{m+q} < \xi_{m+q-1}$ is the dominant singularity of $L_{m,q}(z)$ and

$$L_{m,q}(z) = R_{m,q}(\xi_{m+q}) - C_{m,q}\left(1 - \frac{z}{\xi_{m+q}}\right)^{\frac{1}{2}} + \mathrm{O}\left(\left|1 - \frac{z}{\xi_{m+q}}\right|\right),$$

where

$$C_{m,q} = \frac{\xi_{m+q}^{cq-d}\sqrt{\xi_{m+q}^{a+d}\sum_{j=0}^{m+q-1}(a+d+bj)\xi_{m+q}^{bj}}}{\prod_{i=0}^{q-1}\delta_{m+i}(\xi_{m+q})}.$$

Corollary

Then the number of m-open lambda terms with exactly q abstractions and size n is

$$L_{m,q,n} \sim C_{m,q}\frac{\xi_{m+q}^{-n}}{2\sqrt{\pi n^3}}, \quad \text{as} \quad n \to \infty.$$

Boltzmann sampling

Singular Boltzmann output size according to Boltzmann distribution

$$\mathbb{P}N = n = \frac{a_n x \rho^n}{A(\rho)}$$

where $A(\rho)$ is the generating function and ρ its dominant singularity.

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The sampler: Construct superclass $\mathcal{L}_{0,N} \supseteq \mathcal{L}_0$ tending to $\mathcal{L}_0$ and reject unwanted results.

$$\begin{cases} L_{N,0} & = zL_{N,1} + zL_{N,0}^2, \\ L_{N,1} & = zL_{N,2} + zL_{N,1}^2 + z, \\ L_{N,2} & = zL_{N,3} + zL_{N,2}^2 + z + z^2 \quad, \\ \dots & = \dots, \\ L_{N,N-1} & = zL_{N,N} + zL_{N,N-1}^2 + z\dfrac{1 - z^{N-1}}{1 - z}, \\ L_{N,N} & = zL_{N,N} + zL_{N,N}^2 + \dfrac{z}{1 - z} \end{cases}$$

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Rejection if de Bruijn index larger than N drawn in ΓL_N.

Boltzmann sampling

Extra rejection:

- Costs bounded by object size

Boltzmann sampling

Extra rejection:

- Costs bounded by object size
- Unwanted terms are open terms in $\mathcal{L}_{0,N}$:

$$\frac{[z^n]L_0(z)}{[z^n]L_{N,0}(z)} \longrightarrow 1.$$

Speed is exponential: For $N = 20$, the proportion of closed terms is 0.999999998.

Boltzmann sampling

Experiments with $N = 20$:

$$\mathbb{P}\{\text{unary}\} \approx 0.2955977425, \qquad \mathbb{P}\{\text{binary}\} = \mathbb{P}\{\text{leaf}\} \approx 0.3522011287.$$

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$X_N := \#$ leaves in $\mathcal{L}_{N,0} = \#$ leaves in $\mathcal{L}_{N,N}$.

Thus, all moments of X_N and $X := \#$ leaves in a closed term are asymptotically equal.

Theorem

Let X_n the number of variables in a lambda-term of size n. Then

$$X_n \sim \mathcal{N}(\mu n, \sigma^2 n)$$

where $\mu = \sigma^2 \approx \alpha$ *with* $\alpha = \dfrac{1-\rho}{2} = 0.3522011287\ldots$.

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Sampler has same complexity as sampler for trees (linear in approximate size)

On laptop with CPU i7-5600U, clock rate 2.6 GHz, it is possible to draw a lambda-term of size in the range [1 000 000, 2 000 000] in less than 10 minutes.

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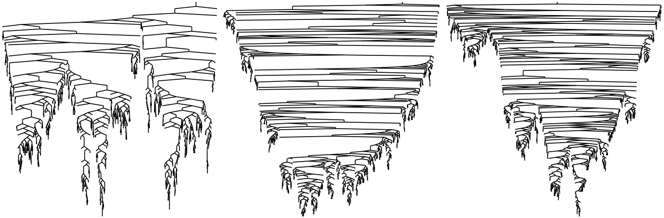

Figure: Three uniform random lambda-terms of size 2098, 2541, 2761.

Related work and perspectives

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- restricting the number of variables bound by an abstraction
  cf. Bodini, Gardy, Jacquot 2010 (BCI/BCK), Bodini, Gardy, G., Jacquot 2013(gen. BCI), Bodini, G. 2014 (BCK_2)
  - Shape characteristics of BCI/BCK/gen BCI terms (G., Larcher, in progress)

Thank you!