Algebraic logic of paths[1]

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[1]Thanks to: Maria João Gouveia (ULisboa), Srecko Brlek (UQAM), Daniela Muresan (UCagliari, UBucarest)

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Goals

Explore connections between Logic and Combinatorics.

- Logic of *provability*:
  mainly ordered algebraic structures related to logic
  (Heyting algebras, residuated lattices, quantales . . .)
- Combinatorics:
  of words, bijective, enumerative, . . . a bit of geometry, as well.

Thesis:

- it is relevant,
- it is fun.

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- L. Santocanale and M. J. Gouveia. The continuous weak order. Submitted. Dec. 2018. Link to Hal.
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Plan

Permutations, words, paths

The quantaloid of discrete paths

Adding the continuum

The continuous Bruhat order

Idempotents, a dive into enumerative combinatorics

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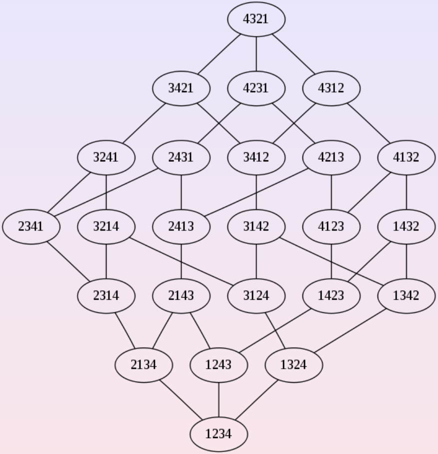
The weak Bruhat order, i.e. the permutohedron P(n)



Theorem (Santocanale & Wehrung, 2018)

The equational theory of the lattices P(n) *is decidable and non-trivial.*

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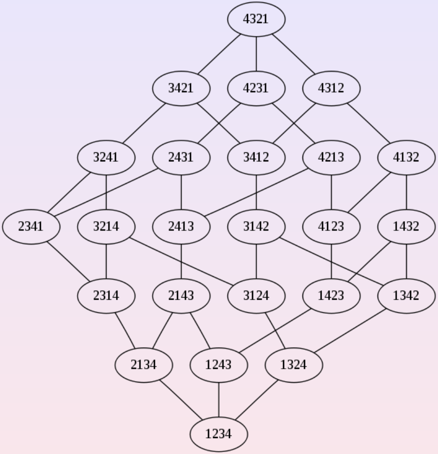

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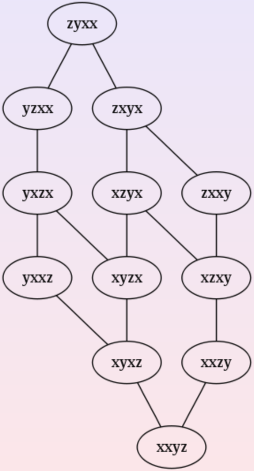
The multinomial lattice $P(n_1, n_2, \ldots, n_d)$



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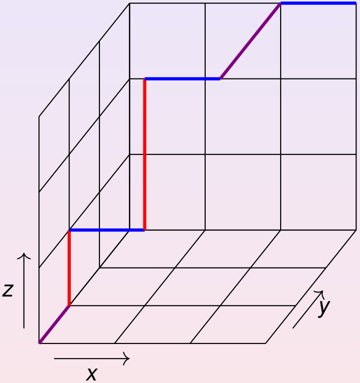

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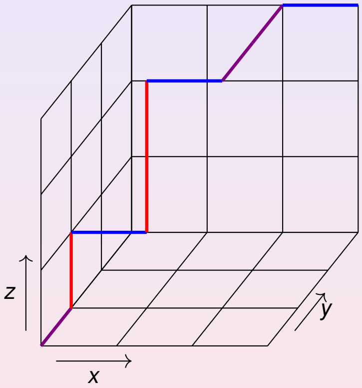
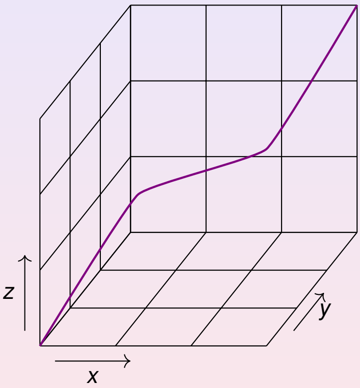
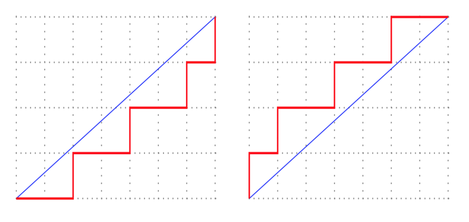




Motivations: discrete geometry and Christoffel words



Christoffel words are images of the diagonal via right/left adjoints:

Are there generalizations of these ideas in dimensions ≥ 3?

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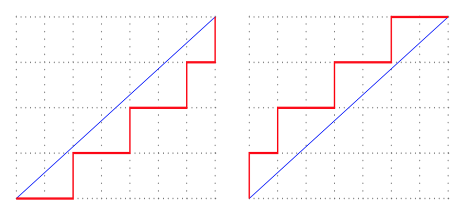

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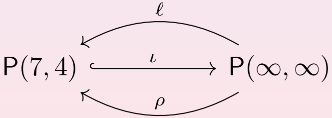

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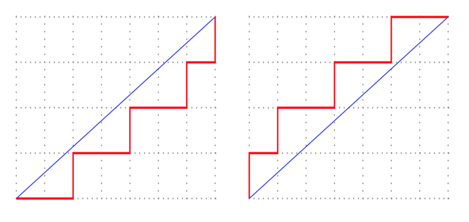

Christoffel words are images of the diagonal via right/left adjoints:

$$\mathsf{P}(7,4) \overset{\iota}{\hookrightarrow} \mathsf{P}(\infty,\infty), \qquad \ell, \rho : \mathsf{P}(\infty,\infty) \to \mathsf{P}(7,4)$$

Are there generalizations of these ideas in dimensions ≥ 3?

Plan

A category P of words/discrete-paths

- Objects : natural numbers $0, 1, \ldots, n, \ldots$
- Arrows:

$$\mathsf{P}(n, m) := \{ w \in \{ x, y \}^* \mid |w|_x = n, |w|_y = m \}$$

- Composition:

$xyxyyx \otimes yxxyxy$:

$$\begin{array}{ccccccc} \epsilon & y & xx & y & x & y & \epsilon \\ & \big| & & \big| & & \big| & \\ \epsilon & x & y & x & yy & x & \epsilon \end{array} \quad \rightsquigarrow \quad \epsilon|xxy|xyy|\epsilon \quad \rightsquigarrow \quad xxyxyy$$

The standard bijection(s)

Let $[n] := \{1, \ldots, n\}$, $\mathbb{I}_n := \{0, 1 \ldots, n\}$.

Standard bijection (cf. also compositions of n):

$xxyxyyxxy \in \mathrm{P}(5, 4)$:

$f : [5] \longrightarrow \mathbb{I}_4$:

$$f(1) = f(2) = 0$$
$$f(3) = 1$$
$$f(4) = f(5) = 3$$

That is:

$$\mathrm{P}(n, m) \simeq Pos([n], \mathbb{I}_m) \simeq \mathrm{SLat}_{\vee}(\mathbb{I}_n, \mathbb{I}_m)\,.$$

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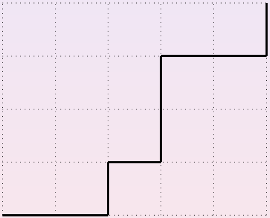

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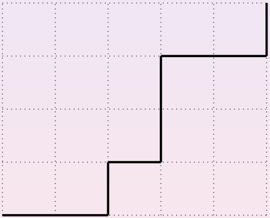

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- The correspondence

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- Under the bijection, composition is function composition.
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Counting factorizations

$$\binom{n+m}{n}\binom{m+k}{k} = \sum_{i=0}^{m}\binom{n+m+k-i}{m-i}\binom{n}{i}\binom{k}{i}$$

In particular

$$\binom{2n}{n}^2 = \sum_{i=0}^{n}\binom{3n-i}{n-i}\binom{n}{i}^2 .$$

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Properties of P

- $\mathsf{P}(n, m)$ is a finite distributive lattice (under the dominance ordering),
- ...whence, an Heyting algebra (algebraic model of Intuitionist Logic).
- $\mathsf{P}(n, n)$ is a *non-commutative quantale/involutive residuated lattice* (algebraic model of non-commutative cyclic classical linear logic):

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\begin{gathered}
(\bigvee_i w_i) \otimes (\bigvee_j w_j) = \bigvee_{i,j} w_i \otimes w_j \,, \\
w_1 \otimes w_2 \leq w_3 \quad \text{iff} \quad w_2 \leq w_1 \multimap w_3 \quad \text{iff} \quad w_1 \leq w_3 \multimapinv w_2 \,, \\
(w^\star)^\star = w \,, \\
w_1 \oplus w_2 := (w_2^\star \otimes w_1^\star)^\star \,, \\
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$$w_1 \otimes w_2 \leq w_3 \quad \text{iff} \quad w_2 \leq w_1 \multimap w_3 \quad \text{iff} \quad w_1 \leq w_3 \circ\!\!- w_2 \,,$$

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Properties of P

More generally:

- P is a quantaloid (sup-lattice enriched):

$$\mathsf{P}(n,m) \simeq \mathrm{SLat}_{\bigvee}(\mathbb{I}_n, \mathbb{I}_m)\,.$$

- The correspondence

$$f \mapsto f^{\wedge}\,, \qquad\qquad f^{\wedge}(x) := \bigwedge_{x<y} f(y)\,,$$

yields isomorphisms

$$\mathrm{SLat}_{\bigvee}(\mathbb{I}_n, \mathbb{I}_m) \simeq \mathrm{SLat}_{\bigwedge}(\mathbb{I}_n, \mathbb{I}_m) \simeq \mathrm{SLat}_{\bigvee}^{op}(\mathbb{I}_m, \mathbb{I}_n)\,.$$

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$\star$-autonomous structure

$$f^{\star} := \text{left-adjoint-of}(f^{\wedge}) \quad (= \ (\text{right-adjoint-of}(f))^{\vee} \quad).$$

On words: exchanges *x*s and *y*s.

That is:

Proposition

P *is a $\star$-autonomous quantaloid.*

Dual composition:

$$g \oplus f := (f^{\star} \circ g^{\star})^{\star} = (g^{\wedge} \circ f^{\wedge})^{\vee}.$$

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Word reconstruction problem

For which triple

$$w_{1,2} \in \mathrm{P}(3,2)_{x,y},\ w_{2,3} \in \mathrm{P}(2,4)_{y,z},\ w_{1,3} \in \mathrm{P}(3,4)_{x,z}\,,$$

there exists word $w \in \mathrm{P}(3,2,4)_{x,y,z}$ such that:

$$w_{1,2} = w\restriction_{x,y}\,, \quad w_{2,3} = w\restriction_{y,z}\,, \quad w_{1,3} = w\restriction_{x,z}\,?$$

A word exists (and is unique) iff $(w_{1,2}, w_{2,3}, w_{1,3})$ satisfies

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Clopen families

Let

$$[d]_2 := \{ (i,j) \mid 1 \leq i < j \leq d \},$$

pick

$$(v_1, \ldots, v_d) \in \mathbb{N}^d,$$

and consider

$$w : [d]_2 \longrightarrow \bigcup_{n,m} \mathrm{P}(n,m) \quad \text{such that} \quad w_{i,j} \in \mathrm{P}(v_i, v_j).$$

We say that

- *closed* if
$$w_{i,j} \otimes w_{j,k} \leq w_{i,k}, \qquad \text{for each } i < j < k,$$
- *open* if
$$w_{i,k} \leq w_{i,j} \oplus w_{j,k}, \qquad \text{for each } i < j < k,$$
- *clopen* if it is both closed and open.

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The poset of clopens

Standard theory:

- Clopens form a poset: $w \leq w'$ iff $w_{i,j} \leq w'_{i,j}$ $(1 \leq i < j \leq d)$
- Closed (resp., open) tuples form a lattice.

Proposition

Clopens form a lattice as well.

Proof. Use the rule MIX:

$$g \otimes f \leq g \oplus f\,.$$

Proposition

Clopens bijectively correspond to

- *maximal chains in the product lattice* $\prod_{i=1,\ldots,d} \mathbb{I}_{v_i}$,
- *words in the multinomial lattice* $\mathrm{P}(v_1, \ldots, v_d)$.

Under this bijection, the lattice of clopens is the multinomial lattice $\mathrm{P}(v_1, \ldots, v_d)$.

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Algebraic remarks

The construction of the multinomial lattices $P(v_1, \ldots, v_d)$ only depends on the algebraic properties of the quantaloid P.

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For every $\star$-autonomous quantale Q satisfying MIX (and each $d \geq 3$), the poset of clopens $Q(d)$ is a lattice.

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For every $\star$-autonomous quantaloid Q satisfying MIX, each $d \geq 3$, and $(v_1, \ldots, v_d) \in Obj(Q)$, the poset $Q(v_1, \ldots, v_d)$ of clopens is a complete lattice.

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For every $\star$-autonomous quantaloid Q satisfying MIX, each $d \geq 3$, and $(v_1, \ldots, v_d) \in Obj(\mathsf{Q})$, the poset $\mathsf{Q}(v_1, \ldots, v_d)$ of clopens is a complete lattice.

Algebraic remarks

The construction of the multinomial lattices $\mathsf{P}(v_1, \ldots, v_d)$ only depends on the algebraic properties of the quantaloid P.

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Plan

A larger category P_+ of words/paths

- Objects: extended natural numbers $0, 1, \ldots, n, \ldots, \infty$.
- Arrows: $\mathsf{P}_+(n, m) = \mathrm{SLat}_\vee(\mathbb{I}_n, \mathbb{I}_m)$, where

$$\mathbb{I}_\infty := [0, 1]\,.$$

Join-continuous functions as continuous words

Lemma

Bijection/equality between the following kind of data:

- *maximal chains in* $[0,1]^2$,
- *images of continuous monotone functions* $\pi : [0,1] \to [0,1]^2$ *preserving endpoints,*
- *join-continuous (or meet-continuous) functions from* $[0,1]$ *to* $[0,1]$.

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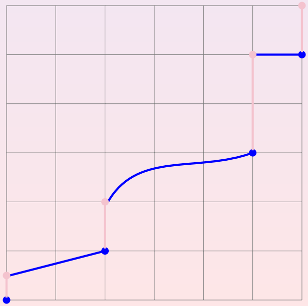

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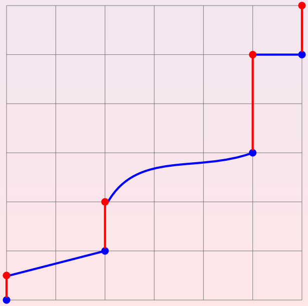

Generalized results

Proposition

P_+ *is a* $\star$*-autonomous quantaloid (satisfying mix:* $\otimes \leq \oplus$*).*

Let $\vec{v} = (v_1, \ldots, v_d)$ with $v_i \in \mathbb{N} \cup \{\infty\}$, so $v : [d] \to (\mathsf{P}_+)_0$.

Proposition

Clopens over $\vec{v}$ *bijectively correspond to maximal chains in the product lattice* $\prod_{i=1,\ldots,n} \mathbb{I}_{v_i}$*. Therefore, these maximal chains can be ordered so they form a lattice.*

Remark. Bijection/equality between the following kind of data:

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Plan

The continuous Bruhat order of dimension d

- The lattice structure of $\mathsf{P}_+(\vec{\omega})$, $\vec{\omega} := (\underbrace{\infty, \ldots, \infty}_{d-\text{times}})$,
- For every $\vec{v} \in \mathbb{N}^d$ and every collection of lattice embeddings $\iota = \{\, \mathbb{I}_{v_i} \to \mathbb{I}_\infty \mid i = 1, \ldots, d \,\}$, there is a lattice embedding

$$P(\vec{v}, \iota) : \mathsf{P}(\vec{v}) \longrightarrow \mathsf{P}_+(\vec{\infty})$$

- $\mathsf{P}_+(\vec{\infty})$ is the Dedekind-MacNeille completion of the colomit of these embeddings.

Generation and discrete approximations

- Canonical cocone ι_v, with $\iota_{v_i}(k) = \frac{k}{v_i}$.
- $\mathsf{P}_+(\vec{\infty})$ is a $\bigvee\bigwedge$-completion of the colomit of the $\mathsf{P}(\vec{v})$.
- The diagonal lives in $\mathsf{P}_+(\vec{\infty})$, it is a join of elements of thos colimit.
- Open problem: characterize those elements from $\mathsf{P}_+(\vec{\infty})$ that are a join of elements of this colimit.

Open problems

- determine the largest class of chains extending P into a $\star$-autonomous quantaloid ...
- equational theories of the lattices $\mathsf{P}(\vec{v})$, $\vec{v} \in \mathbb{N}^d$,
- equational theories of the residuated lattices $\mathsf{P}(n, n)$, $n = 0, 1, \ldots \infty$,
- congruences of the residuated lattices $\mathsf{P}(n, n)$,
- ... and their idempotents (actually, not so open, see next slides),
- ...

Plan

Idempotents as emmentalers[3]

Definition

Le A be a complete join-semilattice. An emmentaler on A is a collection $\{ [y_i, x_i] \mid i \in I \}$ of pairwise disjoint intervals of A such that

- $\{ y_i \mid i \in I \}$ closed under meets,
- $\{ x_i \mid i \in I \}$ closed under joins.

Lemma

Let A be a complete join-semilattice, let $f \in \mathrm{SLat}_{\bigvee}(A, A)$ be idempotent, and let $f \dashv g$. Then $\{ [f(x), g(f(x))] \mid x \in A \}$ is an emmentaler of A.

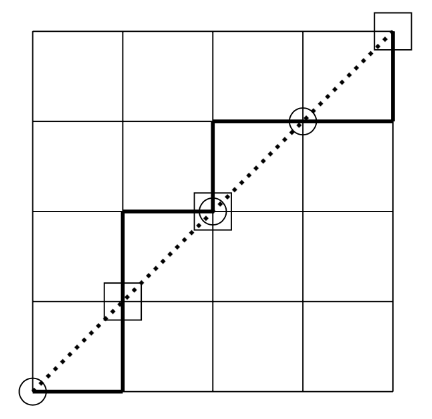
This sets up a bijective correspondence between idempotents and emmentalers.

[3] Thanks to Daniela Muresan

An emmentaler on $\mathbb{I}_n$

. . . is a sequence

$$0 = y_0 \leq x_0 < y_1 \leq x_1 < \ldots y_k \leq x_k = n$$



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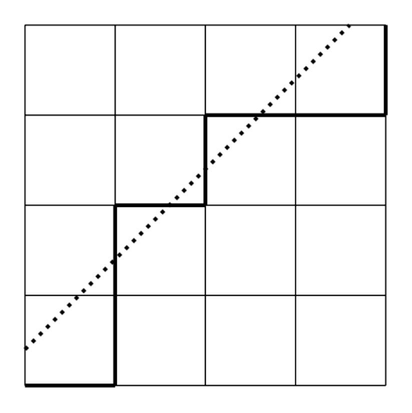

Characterizations of idempotent paths

- Bijection with words $w \in \{1, -1, 0\}^*$, $|w| = n$, w avoids $-10^* - 1$,
- Geometric characterization:
  Every NE-turn is above $y = x + \frac{1}{2}$, every EN-turn is below this line.

Let f_n be the sequence of Fibonacci numbers.

Proposition

The number of idempotents in $\mathrm{SLat}_{\bigvee}(\mathbb{I}_n, \mathbb{I}_n)$ *equals* f_{2n+1}.

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Counting idempotents

Remark:

$$Pos([n],[n]) = \text{strict maps in } \mathrm{SLat}_{\bigvee}(\mathbb{I}_n, \mathbb{I}_n)$$

$Pos([n],[n])$ is a submonoid of $\mathrm{SLat}_{\bigvee}(\mathbb{I}_n, \mathbb{I}_n)$.

Bijective proofs of the following results:

Proposition (Howie 1971)

The number of idempotents in $Pos([n],[n])$ equals f_{2n}.

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Thank you !!!

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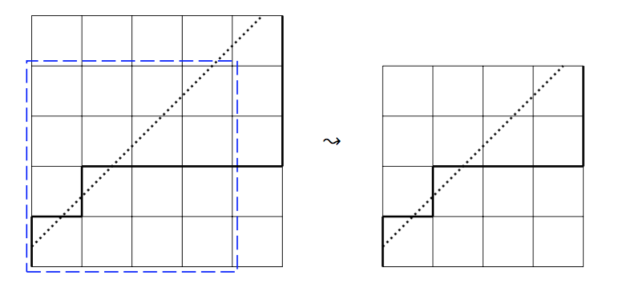

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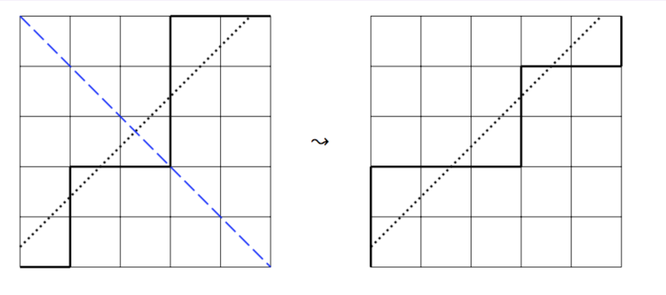

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