Asymptotic enumeration of the linear and affine closed lambda terms with natural size

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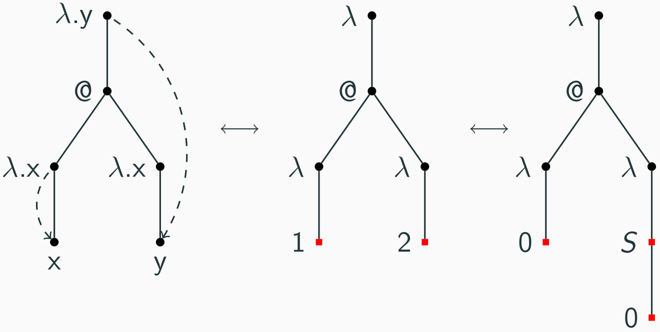
Definition of lambda terms

$$T ::= x \,|\, \lambda x.T \,|\, T\ T \quad \to \quad T ::= n \,|\, \lambda M \,|\, M\ M$$

$\lambda x.T$: *abstraction*, *unary node* $\qquad$ $(T\ T)$: *application*, *binary node*

$$\lambda y.((\lambda x.x)(\lambda x.y)) \leftrightarrow \lambda(\lambda 1 \lambda 2) \leftrightarrow \lambda((\lambda 0)(\lambda(S0)))$$



Notion of size of a λ-term

- variable size equals to 0,
- variable size equals to 1,
- variable size depends on its de Bruijn index.

Notion of size of a λ-term

$$
\begin{aligned}
|0| &= a \\
|Sn| &= |n| + b \\
|\lambda M| &= |M| + c \\
|MN| &= |M| + |N| + d.
\end{aligned}
$$

- $a = b = c = d = 1$ - natural size [Bendkowski, Grygiel, Lescanne, Zaionc (2016)],
- $b = 1, a = c = d = 2$ - binary lambda calculus [Tromp (2006)].

Asymptotic number of closed λ-terms in natural size

Theorem (Bodini, Gittenberger, G. (2018))

Let l_n denote the number of closed λ-terms of natural size n. Then

$$l_n \sim C n^{-\frac{3}{2}} \rho^{-n} \quad as \quad n \to \infty,$$

where $\rho = \mathrm{RootOf}\{-1 + 3x + x^2 + x^3\} \approx 0.295598\ldots$, $\rho^{-1} \approx 3.38298\ldots$ *and* $0.077909995266 \leq C \leq 0.077909998229$.

Linear and affine λ-terms

- A **linear** **closed** λ-term (BCI) is a λ-term in which each variable occurs exactly once and there are no free variables.

- An **affine** **closed** λ-term (BCK) is a λ-term in which each variable occurs at most once and there are no free variables.

Linear and affine λ-terms

- density of linear (BCI) among affine (BCK) λ-terms [Grygiel, Idziak, Zaionc (2013)]
- asymptotic number of the linear/affine closed λ-terms with variable size $0, 1$:
  - [Bodini, Gardy, Jacquot (2013)] - bijection with the combinatorial maps
  - [Bodini, Gardy, Gittenberger, Jacquot (2013)] - a growth process of a λ-term via a functional equation for $BCI(k)$ terms
  - [Bodini, Gittenberger (2014)] - study of a functional equation for $BCK(2)$ terms
- recursive equations for the number of linear and affine closed λ-terms with variable size $0, 1$ and natural size:
  - [Lescanne (2018)] - use of *SwissCheese* data structure

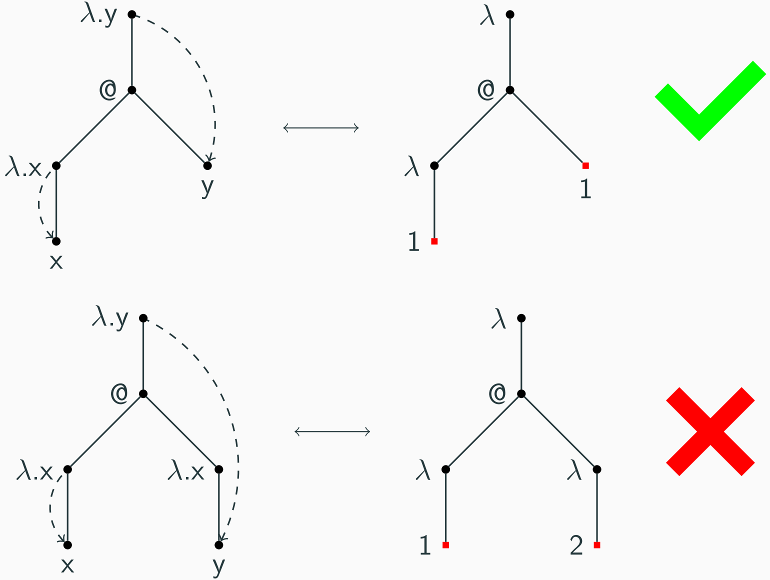
Linear and affine λ-terms with natural size

In [Lescanne (2018)]

$$L(z, \mathbf{u}) = u_0 + zL(z, \mathbf{u})^2 + \sum_{j=1}^{\infty} z^j \frac{\partial L(z, \mathit{tail}(\mathbf{u}))}{\partial u_j},$$

where $\mathbf{u} = (u_0, u_1, u_2, \ldots)$ and $\mathit{tail}(\mathbf{u}) = (u_1, u_2, \ldots)$.

Linear λ-terms in natural size and de Bruijn indices equals 1



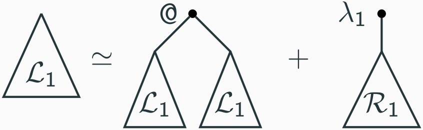
Linear λ-terms in natural size and de Bruijn indices equals 1

- $\mathcal{L}_{k,n}$ denote a set of the linear λ-terms of natural size n and de Bruijn indices less or equal to k
- $l_{k,n} = |\mathcal{L}_{k,n}|$ and $L_k(z) = \sum_n l_{k,n} z^n$

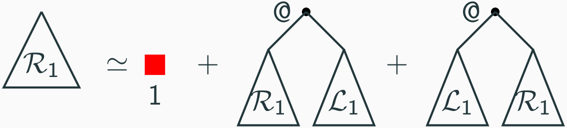
Fact

If $t \in \mathcal{L}_{1,n}$ has p variables then $|t| = n = 3p - 1$.

Linear λ-terms in natural size and de Bruijn indices equals 1



$$L_1(z) = zL_1(z)^2 + zR_1(z)$$



$$R_1(z) = z + 2zL_1(z)R_1(z) \quad \rightarrow \quad R_1(z) = \frac{z}{1 - 2zL_1(z)}$$

Linear λ-terms in natural size and de Bruijn indices equals 1

Lemma

Let $L_1(z)$ denote the OGF of closed linear λ-terms in natural size and de Bruijn indices equal to 1. Then

$$L_1(z) = zL_1(z)^2 + \frac{z^2}{1 - 2zL_1(z)},$$

$$L_1(z) = \frac{1}{2}\left(\frac{1}{\sqrt[3]{3}\sqrt[3]{18z^6 + \sqrt{3}\sqrt{108z^{12} - z^6}}} + \frac{\sqrt[3]{18z^6 + \sqrt{3}\sqrt{108z^{12} - z^6}}}{3^{2/3}z^2} + \frac{1}{z}\right),$$

$$L_1(z) = z^2 + 3z^5 + 16z^8 + 105z^{11} + 768z^{14} + 6006z^{17} + 49152z^{20} + \dots$$

The sequence $1, 3, 16, 105, 768, 6006, 49152, \dots$ appears in OEIS as A085614: The number of elementary arches of size n.

Linear λ-terms in natural size and de Bruijn indices equals 1

Lemma

The number of closed linear λ-terms of natural size n and de Bruijn indices equal to 1 satisfies

$$l_{1,n} \underset{n\to\infty}{\sim} \begin{cases} \frac{3}{2^{\frac{7}{6}}\sqrt{\pi n^3}} \left(2^{\frac{1}{3}}\sqrt{3}\right)^n & \text{if } (n+1)|3 \\ 0 & \text{otherwise.} \end{cases}$$

Linear λ-terms in natural size and de Bruijn indices equals 1

Lemma

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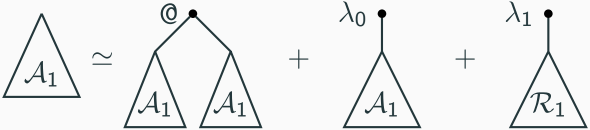
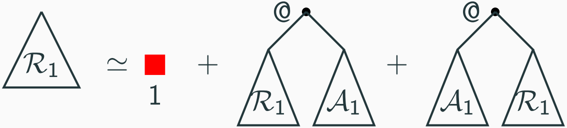
$$l_{1,n} \underset{n\to\infty}{\sim} \begin{cases} \frac{3}{2^{\frac{7}{6}}\sqrt{\pi n^3}} \left(2^{\frac{1}{3}}\sqrt{3}\right)^n & \textit{if } (n+1)|3 \\ 0 & \textit{otherwise.} \end{cases}$$

$$\rho_{L_1}^{-1} = 2^{1/3}\sqrt{3} \approx 2.18225\dots \quad \textit{and} \quad \rho^{-1} \approx 3.38298\dots$$

Affine λ-terms in natural size and de Bruijn indices equals 1



$$A_1(z) = zA_1(z)^2 + zA_1(z) + zR_1(z)$$



$$R_1(z) = z + 2zA_1(z)R_1(z) \quad \rightarrow \quad R_1(z) = \frac{z}{1 - 2zA_1(z)}$$

Affine λ-terms in natural size and de Bruijn indices equals 1

Lemma

Let $A_1(z)$ denote the OGF of closed affine λ-terms in natural size and de Bruijn indices equal to 1. Then

$$A_1(z) = zA_1(z)^2 + zA_1(z) + \frac{z^2}{1 - 2zA_1(z)},$$

$$\begin{aligned} A_1(z) = & -\frac{2z^2 - 3z}{6z^2} \\ & + \frac{\sqrt[3]{46z^6 + 18z^5 - 9z^4 + 3\sqrt{3}\sqrt{76z^{12} + 72z^{11} - 40z^{10} + 12z^9 - 13z^8 + 6z^7 - z^6}}}{6z^2} \\ & - \frac{-4z^4 + 6z^3 - 3z^2}{6z^2\sqrt[3]{46z^6 + 18z^5 - 9z^4 + 3\sqrt{3}\sqrt{76z^{12} + 72z^{11} - 40z^{10} + 12z^9 - 13z^8 + 6z^7 - z^6}}}, \end{aligned}$$

$$A_1(z) = z + z^2 + z^3 + z^4 + 4z^5 + 8z^6 + 13z^7 + 35z^8 + 84z^9 + 172z^{10} + \dots$$

Affine λ-terms in natural size and de Bruijn indices equals 1

Lemma

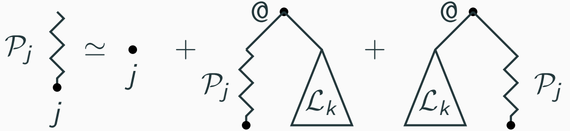
The number of closed affine λ-terms of natural size n and de Bruijn indices equal to 1 satisfies

$$a_{1,n} \underset{n\to\infty}{\sim} C_{A_1} n^{-\frac{3}{2}} \rho_{A_1}^{-n},$$

where
$\rho_{A_1} = RootOf[-1 + 6x - 13x^2 + 12x^3 - 40x^4 + 72x^5 + 76x^6] \approx 0.372288\ldots$ *and* $C_{A_1} \approx 0.31462\ldots$

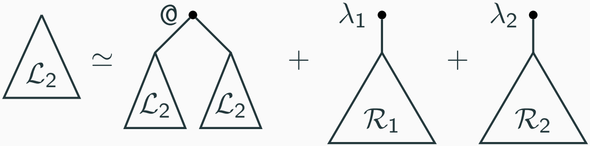
Linear λ-terms in natural size and de Bruijn indices ≤ 2

Let $\mathcal{P}_j$ denote a sub-term that is a path of left-right $\mathcal{L}_k$ terms that finishes with a string of nodes of length j.

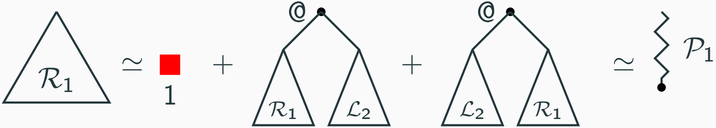


$$P_j(z) = z^j + 2zL_k(z)P_j(z) \quad \rightarrow \quad P_j(z) = \frac{z^j}{1 - 2zL_k(z)}$$

Linear λ-terms in natural size and de Bruijn indices ≤ 2

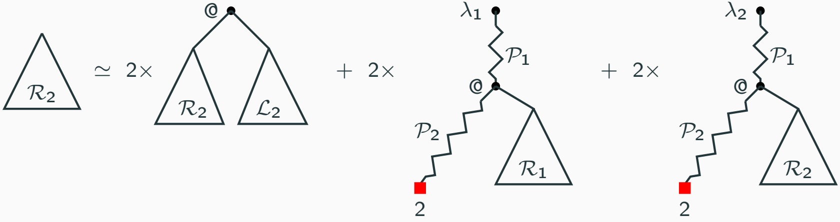


$$L_2(z) = zL_2(z)^2 + zR_1(z) + zR_2(z)$$



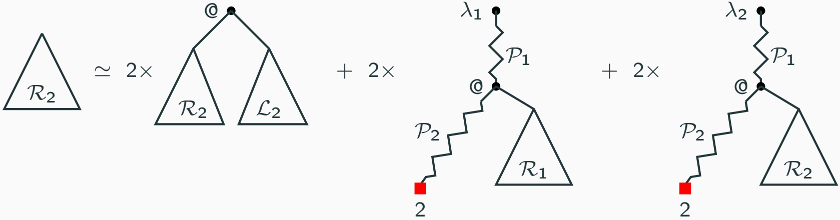
$$R_1(z) = z + 2zR_1(z)L_2(z) \quad \rightarrow \quad R_1(z) = \frac{z}{1 - 2zL_2(z)} = P_1(z)$$

Linear λ-terms in natural size and de Bruijn indices ≤ 2



$$R_2(z) = 2zP_1(z)^3(R_1(z) + R_2(z))$$

Linear λ-terms in natural size and de Bruijn indices ≤ 2



$$R_2(z) = 2zP_1(z)^3(R_1(z) + R_2(z))$$

$$L_2(z) = zL_2(z)^2 + \frac{R_2(z)}{2P_1(z)^3}$$

Linear λ-terms in natural size and de Bruijn indices ≤ 2

Lemma

Let $L_2(z)$ denote the OGF of closed linear λ-terms in natural size and de Bruijn indices less or equal to 2. Then $L_2(z)$ satisfies $P(z, L_2(z)) = 0$ where

$$P(z,y) = 8y^5z^4 - 20y^4z^3 + 18y^3z^2 + y^2\left(2z^5 - 4z^4 - 7z\right) + y\left(-2z^4 + 4z^3 + 1\right) - z^2.$$

The coefficients of $L_2(z)$ are: $0, 0, 1, 0, 0, 3, 2, 0, 16, 24, 4, 105, 252, 108, 776, 2560, 1920, 6390, 25756, 28600, 59552, \ldots$

Linear λ-terms in natural size and de Bruijn indices ≤ 2

Lemma

The number of closed linear λ-terms of natural size n and de Bruijn indices less or equal to 2 satisfies

$$l_{2,n} \underset{n\to\infty}{\sim} C_{L_2} n^{-\frac{3}{2}} \rho_{L_2}^{-n},$$

where
$\rho_{L_2} = RootOf[27 + 128x^5 - 3492x^6 - 4104x^7 - 216x^8 - 13824x^{11} + 34560x^{12} - 34560x^{13} + 17280x^{14} - 4320x^{15} + 432x^{16}] \approx 0.418879\ldots$ *and* $C_{L_2} \approx 0.213529\ldots$.

Affine λ-terms in natural size and de Bruijn indices ≤ 2

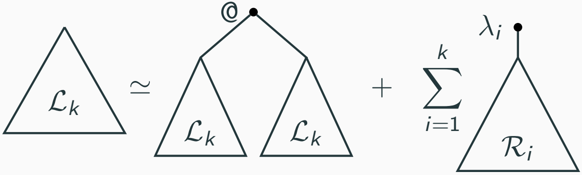
Lemma

The number of closed affine λ-terms of natural size n and de Bruijn indices less or equal to 2 satisfies

$$a_{2,n} \underset{n\to\infty}{\sim} C_{A_2} n^{-\frac{3}{2}} \rho_{A_2}^{-n},$$

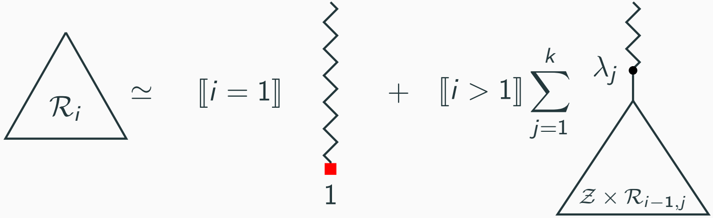
where $\rho_{A_2} = RootOf[27 - 378x + 2295x^2 - 7884x^3 + 18288x^4 - 38140x^5 + 80660x^6 - 133304x^7 + 176904x^8 - 322880x^9 + 532896x^{10} - 502144x^{11} + 621696x^{12} - 638576x^{13} + 197696x^{14} - 697376x^{15} - 12688x^{16} - 338272x^{17} + 1728x^{18}] \approx 0.346216\dots$ *and* $C_{A_2} \approx 0.29841\dots$

Linear λ-terms in natural size and de Bruijn indices $\leq k$



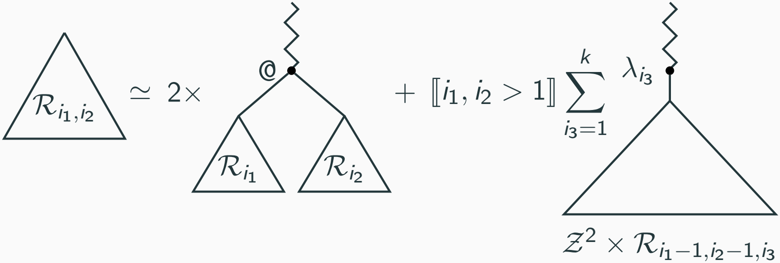
$$L_k(z) = zL_k(z)^2 + z\sum_{i=1}^{k} R_i(z)$$

Linear λ-terms in natural size and de Bruijn indices $\leq k$



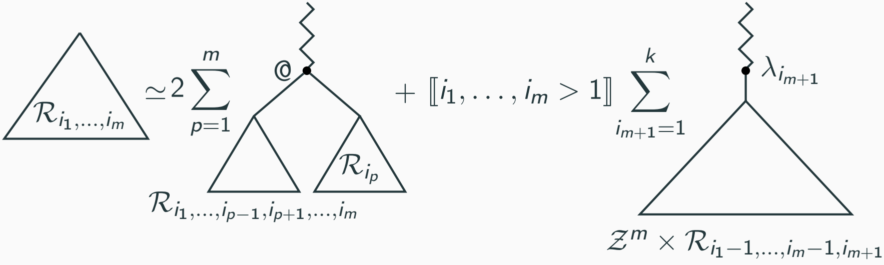
$$R_i(z) = [\![i=1]\!]P_1(z) + [\![i>1]\!]zP_1(z)\sum_{j=1}^{k} R_{i-1,j}(z)$$

Linear λ-terms in natural size and de Bruijn indices $\leq k$



$$R_{i_1,i_2}(z) = 2P_1(z)R_{i_1}(z)R_{i_2}(z) + [\![i_1, i_2 > 1]\!] z^2 P_1(z) \sum_{i_3=1}^{k} R_{i_1-1,i_2-1,i_3}(z)$$

Linear λ-terms in natural size and de Bruijn indices $\leq k$



$$R_{i_1,\ldots,i_m}(z) = 2P_1(z) \sum_{p=1}^{m} R_{i_1,\ldots,i_{p-1},i_{p+1},\ldots,i_m}(z) R_{i_p}(z) +$$

$$[\![i_1, \ldots, i_m > 1]\!] z^m P_1(z) \sum_{i_{m+1}=1}^{k} R_{i_1-1,\ldots,i_m-1,i_{m+1}}(z)$$

Linear λ-terms in natural size and de Bruijn indices ≤ 3

Lemma

The number of closed linear λ-terms of natural size n and de Bruijn indices less or equal to 3 satisfies

$$l_{3,n} \underset{n\to\infty}{\sim} C_{L_3} n^{-\frac{3}{2}} \rho_{L_3}^{-n},$$

where $\rho_{L_3} = RootOf\,[polynomial\ of\ degree\ 287] \approx 0.404569\ldots$ and C_{L_3} is some constant.

Comparison of the exponential terms

k	ρ_{L_k}	$\rho_{L_k}^{-1}$	ρ_{A_k}	$\rho_{A_k}^{-1}$
1	0.458243	2.18225	0.372288	2.68609
2	0.418879	2.38733	0.346216	2.88837
3	0.404569	2.47176	??	??

$$\rho \approx 0.295598\ldots \quad \textit{and} \quad \rho^{-1} \approx 3.38298\ldots$$

Thank you!