Combinatorial operads, rewrite systems, and formal grammars

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Computational Logic and Applications

July 1–2, 2019

Outline

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Operads

Types of algebraic structures

Combinatorics deals with sets (or spaces) of structured objects:

- monoids;
- groups;
- lattices;
- associative alg.;
- Hopf bialg.;
- Lie alg.;
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- dendriform alg.;
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— Example —

The type of monoids can be specified by

1. the operations $\star$ (binary) and $\mathbb{1}$ (nullary);
2. the relations $(x_1 \star x_2) \star x_3 = x_1 \star (x_2 \star x_3)$ and $x \star \mathbb{1} = x = \mathbb{1} \star x$.

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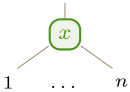

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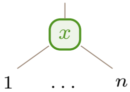

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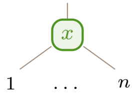

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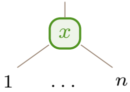

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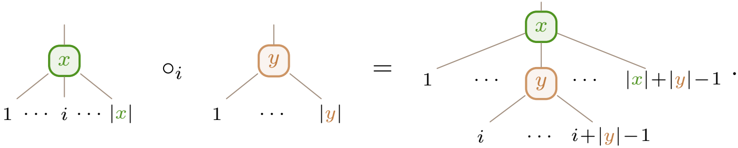

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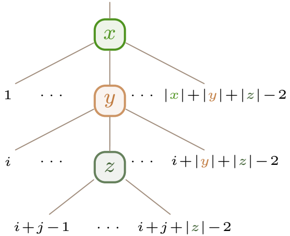
This data has to satisfy some axioms.

Operad axioms

The associativity relation

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$$
$$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$$

says that the pictured operation can be constructed from top to bottom or from bottom to top.



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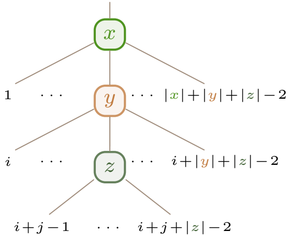

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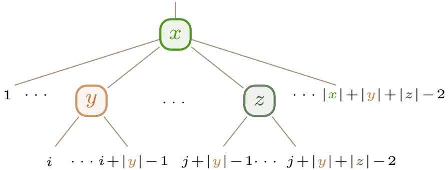

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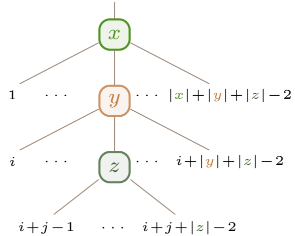

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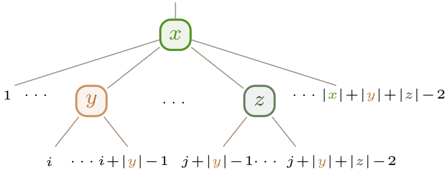
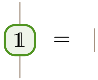


The unitality relation

$$\mathbb{1} \circ_1 x = x = x \circ_i \mathbb{1}$$
$$1 \leqslant i \leqslant |x|$$

says that $\mathbb{1}$ is the identity map.



Operad on permutations

Let $\mathbf{Per}$ be the operad wherein:

- $\mathbf{Per}(n)$ is the set of all permutations of size n, seen through their permutation matrices.

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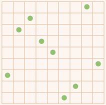

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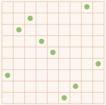

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- The partial composition $\sigma \circ_i \nu$ is the permutation matrix obtained by replacing the ith point of σ by a copy of ν.

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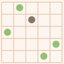
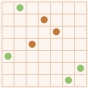
 $\circ_3$  $=$ 

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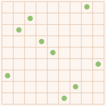

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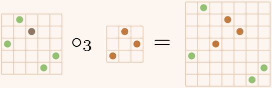

- The unit is the unique permutation  of size 1.

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- **Path**(n) is the set of all paths with n points, that are words $u_1 \dots u_n$ of elements of $\mathbb{N}$.

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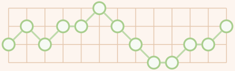

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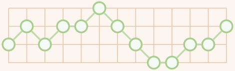

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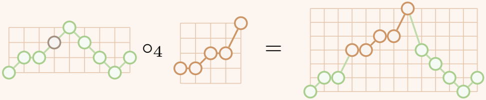

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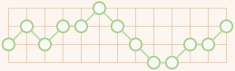

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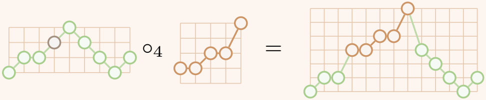

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- The unit is the unique path 0 of size 1, depicted as ○.

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For any $m \geqslant 0$, an m-Dyck path is a path starting and ending with 0 and made of steps 

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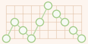 is a 2-Dyck path of size 10.

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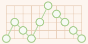 is a 2-Dyck path of size 10.

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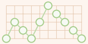 is a 2-Dyck path of size 10.

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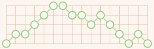 is a Motzkin path of size 16.

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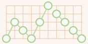 is a 2-Dyck path of size 10.

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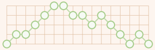 is a Motzkin path of size 16.

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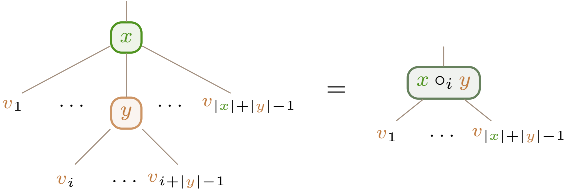

holds for any $x, y \in \mathcal{O}$, $i \in [|x|]$, and $v_1, \ldots, v_{|x|+|y|-1} \in \mathcal{V}$.

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Using infix notation for the binary operation $\star_2$, we obtain the relation

$$(v_1 \star_2 v_2) \star_2 v_3 = v_1 \star_2 (v_2 \star_2 v_3),$$

so that algebras over $\mathbf{As}$ are associative algebras.

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Let $\mathbf{As}$ be the associative operad defined by $\mathbf{As}(n) := \{\star_n\}$ for all $n \geqslant 1$ and $\star_n \circ_i \star_m := \star_{n+m-1}$. This operad is minimally generated by $\star_2$.
Any algebra over $\mathbf{As}$ is a space $\mathcal{V}$ endowed with linear operations $\star_n$ of arity $n \geqslant 1$ where $\star_2$ satisfies, for all $v_1, v_2, v_3 \in \mathcal{V}$,

$$\begin{array}{ccc}(\star_2 \circ_1 \star_2)(v_1, v_2, v_3) & = & \star_2(\star_2(v_1, v_2), v_3) \\ \| & & \| \\ (\star_2 \circ_2 \star_2)(v_1, v_2, v_3) & = & \star_2(v_1, \star_2(v_2, v_3)).\end{array}$$

Using infix notation for the binary operation $\star_2$, we obtain the relation

$$(v_1 \star_2 v_2) \star_2 v_3 = v_1 \star_2 (v_2 \star_2 v_3),$$

so that algebras over $\mathbf{As}$ are associative algebras.

In the same way, there are operads for

- Lie alg.;
- pre-Lie alg. [Chapoton, Livernet, 2001];
- dendriform alg. [Loday, 2001];
- duplicial alg. [Loday, 2008];
- diassociative alg. [Loday, 2001];
- brace alg.

Scope of operads

As main benefits, operads

- offer a formalism to compute over operations;
- allow us to work virtually with all the structures of a type;
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Endowing a set of combinatorial objects with an operad structure helps to

- highlight elementary building block for the objects;
- build combinatorial structures (graded graphs, posets, lattices, *etc.*);
- enumerative prospects and discovery of statistics.

Outline

Enumeration

Syntax trees

An alphabet is a graded set $\mathfrak{G} := \bigsqcup_{n \geqslant 1} \mathfrak{G}(n)$.

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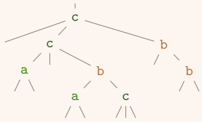
Let $\mathbf{S}(\mathfrak{G})$ be the set of $\mathfrak{G}$-syntax trees, defined recursively by

- $| \in \mathbf{S}(\mathfrak{G})$;
- if $\mathtt{a} \in \mathfrak{G}$ and $\mathfrak{t}_1, \ldots, \mathfrak{t}_{|\mathtt{a}|} \in \mathbf{S}(\mathfrak{G})$, then $\mathtt{a}(\mathfrak{t}_1, \ldots, \mathfrak{t}_{|\mathtt{a}|}) \in \mathbf{S}(\mathfrak{G})$.

— Example —

Let $\mathfrak{G} := \mathfrak{G}(2) \sqcup \mathfrak{G}(3)$ such that $\mathfrak{G}(2) = \{\mathtt{a}, \mathtt{b}\}$ and $\mathfrak{G}(3) = \{\mathtt{c}\}$.



denotes the $\mathfrak{G}$-tree

$$\mathtt{c}(|, \mathtt{c}(\mathtt{a}(|,|),|, \mathtt{b}(\mathtt{a}(|,|), \mathtt{c}(|,|,|))), \mathtt{b}(|, \mathtt{b}(|,|)))$$

having degree 8 and arity 12.

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Let $\mathfrak{t} \in \mathbf{S}(\mathfrak{G})$. Some definitions:

- $|$ is the leaf;
- the degree $\deg(\mathfrak{t})$ of $\mathfrak{t}$ is its number of internal nodes;
- the arity $|\mathfrak{t}|$ of $\mathfrak{t}$ is its number of leaves.

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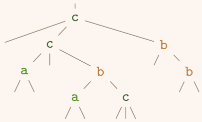

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Compositions of syntax trees

Let $\mathfrak{t}, \mathfrak{s} \in \mathbf{S}(\mathfrak{G})$. For each $i \in [|\mathfrak{t}|]$, the partial composition $\mathfrak{t} \circ_i \mathfrak{s}$ is the tree obtained by grafting the root of $\mathfrak{s}$ onto the ith leaf of $\mathfrak{t}$.

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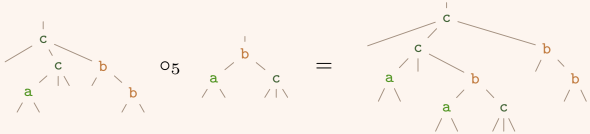

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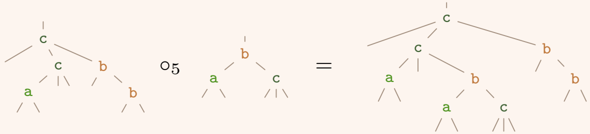
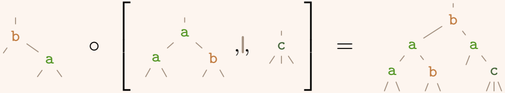


Let $\mathfrak{t}, \mathfrak{s}_1, \dots, \mathfrak{s}_{|\mathfrak{t}|}$ be $\mathfrak{G}$-trees. The full composition $\mathfrak{t} \circ \left[\mathfrak{s}_1, \dots, \mathfrak{s}_{|\mathfrak{t}|}\right]$ is obtained by grafting simultaneously the roots of each $\mathfrak{s}_i$ onto the ith leaf of $\mathfrak{t}$.

— Example —



Free operads

Let $\mathfrak{G}$ be an alphabet.

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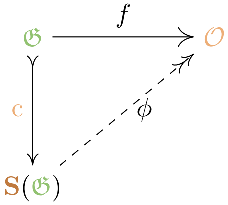
Free operads satisfy the following universality property.

For any alphabet $\mathfrak{G}$, any operad $\mathcal{O}$, and any map $f : \mathfrak{G} \to \mathcal{O}$ preserving the arities, there exists a unique operad morphism $\phi : \mathbf{S}(\mathfrak{G}) \to \mathcal{O}$ such that $f = \phi \circ \mathrm{c}$.



Factors and prefixes

Let $\mathfrak{t}, \mathfrak{s} \in \mathbf{S}(\mathfrak{G})$.

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If $\mathfrak{t}$ decomposes as

$$\mathfrak{t} = \mathfrak{r} \circ_i \left(\mathfrak{s} \circ \left[\mathfrak{r}_1, \ldots, \mathfrak{r}_{|\mathfrak{s}|}\right]\right)$$

for some trees $\mathfrak{r}$, $\mathfrak{r}_1$, …, $\mathfrak{r}_{|\mathfrak{s}|}$, and $i \in [|\mathfrak{r}|]$, then $\mathfrak{s}$ is a factor of $\mathfrak{t}$.

This property is denoted by $\mathfrak{s} \preccurlyeq_{\mathrm{f}} \mathfrak{t}$.

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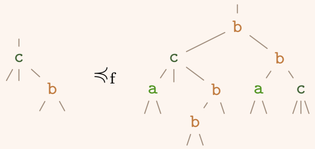

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If in the previous decomposition $\mathfrak{r} = |$, then $\mathfrak{s}$ is a prefix of $\mathfrak{t}$.

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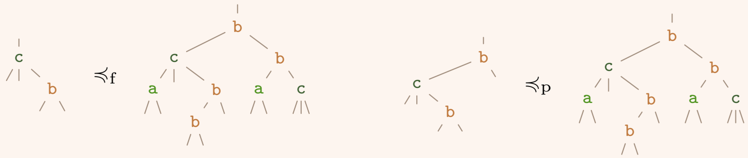

Pattern avoidance and enumeration

A $\mathfrak{G}$-tree $\mathfrak{t}$ avoids a $\mathfrak{G}$-tree $\mathfrak{s}$ if $\mathfrak{s} \not\preccurlyeq_{\mathrm{f}} \mathfrak{t}$.

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$$\mathrm{A}(\mathcal{P}) = \{\mathfrak{t} \in \mathbf{S}(\mathfrak{G}) : \text{ for all } \mathfrak{s} \in \mathcal{P}, \mathfrak{s} \not\preccurlyeq_{\mathrm{f}} \mathfrak{t}\}.$$

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Enumerate $\mathrm{A}(\mathcal{P})$ w.r.t. the arities of the trees.

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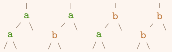
- A () is enumerated by $1, 2, 4, 8, 16, 32, 64, 128, \ldots$.

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- A () is enumerated by $1, 2, 4, 8, 16, 32, 64, 128, \ldots$.
- A () is enumerated by $1, 1, 2, 4, 9, 21, 51, 127, \ldots$ (**A001006**).

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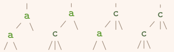
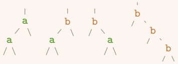
- A () is enumerated by $1, 1, 2, 4, 9, 21, 51, 127, \ldots$ (**A001006**).
- A () is enumerated by $1,2,5,13,35,96,267,750,\ldots$ (**A005773**).

— Question —

Enumerate $\mathrm{A}(\mathcal{P})$ w.r.t. the arities of the trees.

Formal power series of trees

For any $\mathcal{P}, \mathcal{Q} \subseteq \mathbf{S}(\mathfrak{G})$, let

$$\mathbf{F}(\mathcal{P}, \mathcal{Q}) := \sum_{\substack{\mathfrak{t} \in \mathbf{S}(\mathfrak{G}) \\ \mathfrak{t} \in \mathrm{A}(\mathcal{P}) \\ \forall \mathfrak{s} \in \mathcal{Q}, \mathfrak{s} \not\preccurlyeq_{\mathrm{p}} \mathfrak{t}}} \mathfrak{t}.$$

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Since

- $\mathbf{F}(\mathcal{P}, \emptyset)$ is the formal sum of all the trees of $\mathrm{A}(\mathcal{P})$;
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the series $\mathbf{F}(\mathcal{P}, \mathcal{Q})$ contains all the enumerative data about the trees avoiding $\mathcal{P}$.

System of equations

When $\mathfrak{G}$, $\mathcal{P}$, and $\mathcal{Q}$ satisfy some conditions, $\mathbf{F}(\mathcal{P}, \mathcal{Q})$ expresses as an inclusion-exclusion formula involving simpler terms $\mathbf{F}(\mathcal{P}, \mathcal{S}_i)$.

— Theorem —

The series $\mathbf{F}(\mathcal{P}, \mathcal{Q})$ satisfies

$$\mathbf{F}(\mathcal{P}, \mathcal{Q}) = \mathsf{I} + \sum_{\substack{k \geqslant 1 \\ \mathsf{a} \in \mathfrak{G}(k)}} \sum_{\substack{\ell \geqslant 1 \\ \left\{\mathcal{R}^{(1)}, \dots, \mathcal{R}^{(\ell)}\right\} \subseteq \mathfrak{M}((\mathcal{P} \cup \mathcal{Q})_{\mathsf{a}}) \\ (\mathcal{S}_1, \dots, \mathcal{S}_k) = \mathcal{R}^{(1)} \dot{+} \cdots \dot{+} \mathcal{R}^{(\ell)}}} (-1)^{1+\ell} \mathsf{a} \bar{\circ} \left[\mathbf{F}(\mathcal{P}, \mathcal{S}_1), \dots, \mathbf{F}(\mathcal{P}, \mathcal{S}_k)\right].$$

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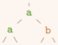
This leads to a system of equations for the generating series of $\mathrm{A}(\mathcal{P})$.

Indeed, the generating series of $\mathrm{A}(\mathcal{P})$ is the series $F(\mathcal{P}, \emptyset)$ where

$$F(\mathcal{P}, \mathcal{Q}) = z + \sum_{\substack{k \geqslant 1 \\ \mathsf{a} \in \mathfrak{G}(k)}} \sum_{\substack{\ell \geqslant 1 \\ \left\{\mathcal{R}^{(1)}, \dots, \mathcal{R}^{(\ell)}\right\} \subseteq \mathfrak{M}((\mathcal{P} \cup \mathcal{Q})_{\mathsf{a}}) \\ (\mathcal{S}_1, \dots, \mathcal{S}_k) = \mathcal{R}^{(1)} \dot{+} \dots \dot{+} \mathcal{R}^{(\ell)}}} (-1)^{1+\ell} \prod_{i \in [k]} F\left(\mathcal{P}, \mathcal{S}_i\right).$$

System of equations

— Example —

For $\mathcal{P} := \{$  $\}$, we obtain the system of formal power series of trees

$$
\begin{aligned}
\mathbf{F}(\mathcal{P}, \emptyset) &= | + \mathtt{a} \bar{\circ} [\mathbf{F}(\mathcal{P}, \{\mathtt{a}\}), \mathbf{F}(\mathcal{P}, \emptyset)] + \mathtt{a} \bar{\circ} [\mathbf{F}(\mathcal{P}, \emptyset), \mathbf{F}(\mathcal{P}, \{\mathtt{b}\})] \\
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\end{aligned}
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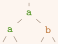
For $\mathcal{P} := \Bigg\{$  $\Bigg\}$, we obtain the system of formal power series of trees

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\end{aligned}
$$

This leads to the system of generating series

$$
\begin{aligned}
F(\mathcal{P}, \emptyset) &= z + F(\mathcal{P}, \{\mathtt{a}\})F(\mathcal{P}, \emptyset) + F(\mathcal{P}, \emptyset)F(\mathcal{P}, \{\mathtt{b}\}) \\
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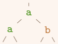
System of equations

— Example —

For $\mathcal{P} :=$ {  }, we obtain the system of formal power series of trees

$$
\begin{aligned}
\mathbf{F}(\mathcal{P},\emptyset) = | &+ \mathtt{a}\bar{\circ}\left[\mathbf{F}(\mathcal{P},\{\mathtt{a}\}),\mathbf{F}(\mathcal{P},\emptyset)\right] + \mathtt{a}\bar{\circ}\left[\mathbf{F}(\mathcal{P},\emptyset),\mathbf{F}(\mathcal{P},\{\mathtt{b}\})\right] \\
&- \mathtt{a}\bar{\circ}\left[\mathbf{F}(\mathcal{P},\{\mathtt{a}\}),\mathbf{F}(\mathcal{P},\{\mathtt{b}\})\right] + \mathtt{b}\bar{\circ}\left[\mathbf{F}(\mathcal{P},\emptyset),\mathbf{F}(\mathcal{P},\emptyset)\right], \\
\mathbf{F}(\mathcal{P},\{\mathtt{a}\}) = | &+ \mathtt{b}\bar{\circ}\left[\mathbf{F}(\mathcal{P},\emptyset),\mathbf{F}(\mathcal{P},\emptyset)\right], \\
\mathbf{F}(\mathcal{P},\{\mathtt{b}\}) = | &+ \mathtt{a}\bar{\circ}\left[\mathbf{F}(\mathcal{P},\{\mathtt{a}\}),\mathbf{F}(\mathcal{P},\emptyset)\right] + \mathtt{a}\bar{\circ}\left[\mathbf{F}(\mathcal{P},\emptyset),\mathbf{F}(\mathcal{P},\{\mathtt{b}\})\right] \\
&- \mathtt{a}\bar{\circ}\left[\mathbf{F}(\mathcal{P},\{\mathtt{a}\}),\mathbf{F}(\mathcal{P},\{\mathtt{b}\})\right].
\end{aligned}
$$

This leads to the system of generating series

$$
\begin{aligned}
F(\mathcal{P},\emptyset) = z &+ F(\mathcal{P},\{\mathtt{a}\})F(\mathcal{P},\emptyset) + F(\mathcal{P},\emptyset)F(\mathcal{P},\{\mathtt{b}\}) \\
&- F(\mathcal{P},\{\mathtt{a}\})F(\mathcal{P},\{\mathtt{b}\}) + F(\mathcal{P},\emptyset)F(\mathcal{P},\emptyset), \\
F(\mathcal{P},\{\mathtt{a}\}) = z &+ F(\mathcal{P},\emptyset)F(\mathcal{P},\emptyset), \\
F(\mathcal{P},\{\mathtt{b}\}) = z &+ F(\mathcal{P},\{\mathtt{a}\})F(\mathcal{P},\emptyset) + F(\mathcal{P},\emptyset)F(\mathcal{P},\{\mathtt{b}\}) \\
&- F(\mathcal{P},\{\mathtt{a}\})F(\mathcal{P},\{\mathtt{b}\}).
\end{aligned}
$$

As a consequence, $F(\mathcal{P},\emptyset)$ satisfies

$$
z - F(\mathcal{P},\emptyset) + (2+z)F(\mathcal{P},\emptyset)^2 - F(\mathcal{P},\emptyset)^3 + F(\mathcal{P},\emptyset)^4 = 0.
$$

Operads and presentations

Let $\mathcal{O}$ be an operad. A congruence of $\mathcal{O}$ is an equivalence relation $\equiv$ on $\mathcal{O}$ preserving the arities and such that $x \equiv x'$ and $y \equiv y'$ imply $x \circ_i y \equiv x' \circ_i y'$ for all $i \in [|x|]$.

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A presentation of $\mathcal{O}$ is a pair $(\mathfrak{G}, \equiv)$ such that $\mathfrak{G}$ is an alphabet and $\equiv$ is a congruence of $\mathcal{O}$ satisfying

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The operad **Motz** admits the presentation $(\mathfrak{G}, \equiv)$ where

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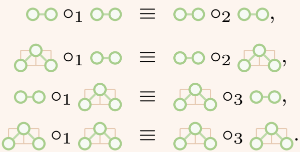
and $\equiv$ is the smallest operad congruence satisfying



Operads and patterns

Let $\mathcal{O}$ be an operad admitting a presentation $(\mathfrak{G}, \equiv)$.

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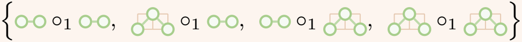

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Rewrite systems on $\mathfrak{G}$-trees are good tools to compute bases (we find terminating and confluent orientations $\Rightarrow$ of $\equiv$).

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Let X be a family of combinatorial objects we want enumerate.

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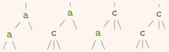
Let a := , c := , and $\mathcal{P}$:= $\Big\{$ 

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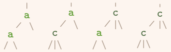
Let $\mathtt{a} :=$ , $\mathtt{c} :=$ , and $\mathcal{P} :=$ $\left\{ \right.$ 

$\left. \right\}$.

We have

$$\mathbf{F}(\mathcal{P}, \emptyset) = | + \mathtt{a} \bar{\circ} [\mathbf{F}(\mathcal{P}, \{\mathtt{a}, \mathtt{c}\}), \mathbf{F}(\mathcal{P}, \emptyset)] + \mathtt{c} \bar{\circ} [\mathbf{F}(\mathcal{P}, \{\mathtt{a}, \mathtt{c}\}), \mathbf{F}(\mathcal{P}, \emptyset), \mathbf{F}(\mathcal{P}, \emptyset)],$$

$$\mathbf{F}(\mathcal{P}, \{\mathtt{a}, \mathtt{c}\}) = |,$$

so that, the generating series of Motzkin paths satisfies

$$F(\mathcal{P}, \emptyset) = z + zF(\mathcal{P}, \emptyset) + zF(\mathcal{P}, \emptyset)^2.$$

Outline

Generation

Context-free grammars

Let $A = V \sqcup T$ be a set where V is a set of variables and T is a set of terminal symbols.

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— Example —

Let $V := \{x, y\}$, $T := \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$, and $\mathcal{R} := \{(x, \mathtt{b}), (x, x\mathtt{a}y), (y, \mathtt{ac})\}$.

We have

$$\mathtt{b}xx \to \mathtt{b}x\mathtt{a}yx \to \mathtt{bba}yx \to \mathtt{bbaac}x.$$

Regular tree grammars

Let V be a set of variables and T be an alphabet of terminal symbols.

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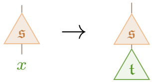

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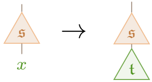

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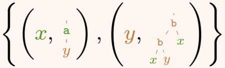
Let $V := \{x, y\}$, $T := \{\mathtt{a}, \mathtt{b}\}$ where $|\mathtt{a}| := 1$, $|\mathtt{b}| := 2$, and $\mathcal{R} :=$  .

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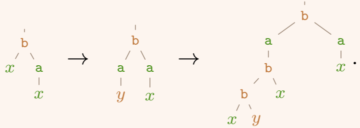

General generation

Objectives:

- Introduce generating systems for any kind of combinatorial objects;
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— Key idea —

Use colored operads, where

- colors play the role of variables and terminal symbols;
- Formal series on colored operad and their operations support enumeration.

Colored operads

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3. $\circ_i$ is a map, called partial composition map,

$$\circ_i : \mathcal{C}(a, u) \times \mathcal{C}\left(u_i, v\right) \to \mathcal{C}\left(a, u \circ_i v\right), \qquad 1 \leqslant i \leqslant |u|,$$

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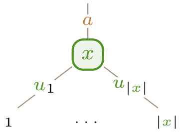
This data has to satisfy some axioms, similar to the ones of operads.

Colored operations

Any element x of $\mathcal{C}(a, u)$ can be seen as a colored operation



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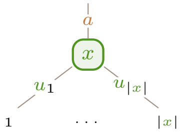
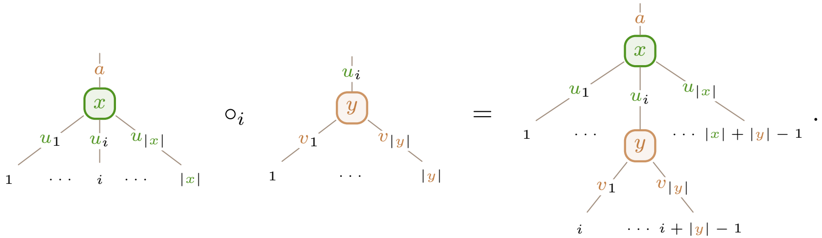


where a color is assigned to the output and to each input of x.

Moreover, the partial composition map requires a condition on the colors:



Bud operads

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The $\mathfrak{C}$-bud operad of $\mathcal{O}$ is the colored operad $\mathbf{B}_{\mathfrak{C}}(\mathcal{O})$ wherein:

- $\mathbf{B}_{\mathfrak{C}}(\mathcal{O})(a, u)$ is the set of all triples (a, x, u) where $x \in \mathcal{O}$ and $(a, u) \in \mathfrak{C} \times \mathfrak{C}^{|x|}$.

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- The colored units are the triples $(c, \mathbb{1}, c)$ where $\mathbb{1}$ is the unit of $\mathcal{O}$.

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— Proposition —

For any set of colors $\mathfrak{C}$, the construction $\mathcal{O} \mapsto \mathbf{B}_{\mathfrak{C}}(\mathcal{O})$ is a functor from the category of operads to the category of colored operads.

Examples of bud operads

The elements of $\mathbf{B}_{\mathfrak{C}}(\mathbf{As})$ are triples $\left(a, \star_{|u|}, u\right)$ where $(a, u) \in \mathfrak{C} \times \mathfrak{C}^{+}$.

— Example —

In $\mathbf{B}_{\{1,2,3\}}(\mathbf{As})$, $(2, \star_4, 3\mathbf{1}12) \circ_2 (1, \star_3, 233) = (2, \star_6, 323312)$.

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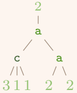
The elements of $\mathbf{B}_{\mathfrak{C}}(\mathbf{S}(\mathfrak{G}))$ are $\mathfrak{C}$-typed $\mathfrak{G}$-syntax trees, that are $\mathfrak{G}$-trees with colors assigned with the root and with each leaf.

— Example —

$\Big(2,$ 

$, 31122\Big) \in \mathbf{B}_{\{1,2,3,4\}}(\mathbf{S}(\{\mathtt{a}, \mathtt{c}\}))$.

This element is drawn as 

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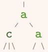
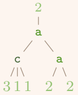
$\left(2,\right.$ 

$\left., 31122\right) \in \mathbf{B}_{\{1,2,3,4\}}(\mathbf{S}(\{\mathtt{a}, \mathtt{c}\}))$.

This element is drawn as 

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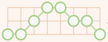
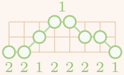
The elements of $\mathbf{B}_{\mathfrak{C}}(\mathbf{Motz})$ are Motzkin paths having a global color and a color assigned with each point.

— Example —

$\left(1,\right.$  $\left., 221222211\right) \in \mathbf{B}_{\mathfrak{C}}(\mathbf{Motz})$.

This element is drawn as 

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A bud generating system is a quintuplet $\mathcal{B} := (\mathcal{O}, \mathfrak{C}, \mathcal{R}, a, T)$ where

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Each element (c, x, u) of $\mathcal{R}$ can be thought as rule having c as left member and u as right member.

Generation

The set $\mathcal{R}$ specifies the rewrite rule $\rightarrow$ on $\mathbf{B}_{\mathfrak{C}}(\mathcal{O})$ by setting

$$x \rightarrow x \circ_i r$$

for any $x \in \mathbf{B}_{\mathfrak{C}}(\mathcal{O})$, $i \in [|x|]$, and $r \in \mathcal{R}$. This is the derivation relation.

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An element x of $\mathbf{B}_{\mathfrak{C}}(\mathcal{O})$ is synchronously generated by $\mathcal{B}$ if

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Generation of particular Motzkin paths

Let the bud generating system $\mathcal{B} := (\mathbf{Motz}, \{1, 2\}, \mathcal{R}, 1, \{1, 2\})$ where

$\mathcal{R} := \big\{(1,$  $, 22)\,,\big(1,$  $, 111\big)\big\}\,.$

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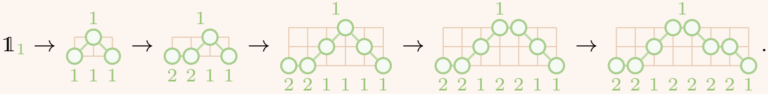
There are in $\mathcal{B}$ the derivations



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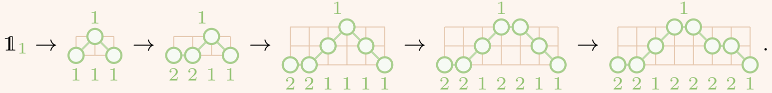

— Proposition —

There is a one-to-one correspondence between the set of Motkzin paths without consecutive  steps and the language of $\mathcal{B}$.

These paths are enumerated by

$$1, 1, 1, 3, 5, 11, 25, 55, 129, 303, 721, 1743, \ldots\ (\mathbf{A104545}).$$

Balanced binary trees

A balanced binary tree is a binary tree $\mathfrak{t}$ such that, for any internal node u of $\mathfrak{t}$, the height of the left subtree and of the right subtree of u differ by at most 1.

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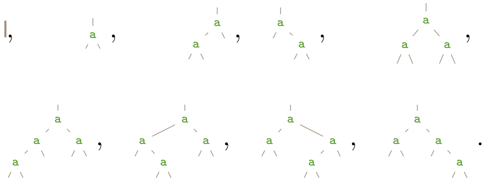

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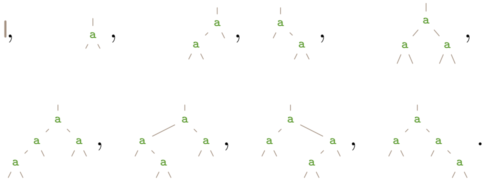

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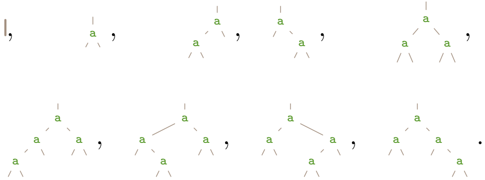

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Their generating series is the specialization $F(x, 0)$ where

$$F(x, y) = x + F\left(x^2 + 2xy, x\right).$$

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Let the bud generating system $\mathcal{B} := (\mathbf{S}(\mathfrak{G}), \{1, 2\}, \mathcal{R}, 1, \{1\})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\mathtt{a}\}$ and

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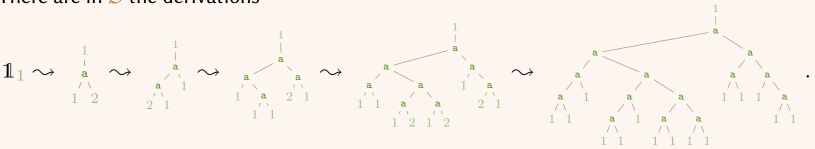

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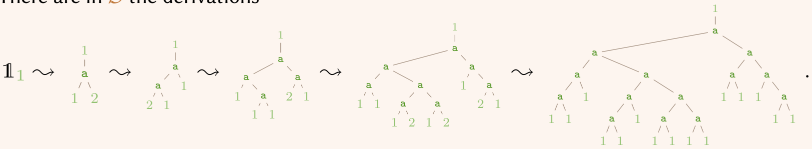

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There is a one-to-one correspondence between the set of balanced binary trees and the synchronous language of $\mathcal{B}$.

Some properties

— Proposition —

For any proper context-free grammar G, there exists a bud generating system $\mathcal{B} := (\mathbf{As}, \mathfrak{C}, \mathcal{R}, a, T)$ such that the language generated by G is in one-to-one correspondance with the language of $\mathcal{B}$.

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For any bud generating system $\mathcal{B}$, the synchronous language of $\mathcal{B}$ is a subset of the language of $\mathcal{B}$.

Random generation

For any $c \in \mathfrak{C}$, let $\mathcal{R}_c$ be the subset of $\mathcal{R}$ of the elements having c as output color.

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Algorithm RBS:

- **Input:**
  1. a bud generating system $\mathcal{B} := (\mathcal{O}, \mathfrak{C}, \mathcal{R}, a, T)$;
  2. An integer $k \geqslant 0$.
- **Output:** an element of the synchronous language of $\mathcal{B}$.

1. **Let** $x := \mathbb{1}_a$;
2. **Repeat** k times:
   2.1 For any $i \in [|x|]$, pick y_i uniformly at random in $\mathcal{R}_c$ where c is the ith input color of x;
   2.2 **Set** $x := x \circ \left[y_1, \ldots, y_{|x|}\right]$;
3. **If** all input colors of x belong to T:
   3.1 **Return** x;
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— Proposition —

If $\mathcal{B} = (\mathcal{O}, \mathfrak{C}, \mathcal{R}, a, T)$ is synchronously unambiguous, the RBS is a uniform random generator of the elements of the synchronous language of $\mathcal{B}$.