The Tamari order for D^3 and derivability in semi-associative Lambek-Grishin Calculus

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Dyck and MIX languages

Dyck A k-dimensional Dyck language, D^k, consists of words over a k-letter alphabet (lexicographically ordered) satisfying the following two constraint:

- MULTIPLICITY: each word contains the k letters with equal frequency
- PREFIX: for every prefix of a word, $\#a_1 \geq \#a_2 \geq \ldots \geq \#a_k$

For example: D^2: language of balanced brackets

MIX respect MULTIPLICITY, drop PREFIX

D_n^k, M_n^k: finite languages with letter multiplicity n

M and D: facts and figures

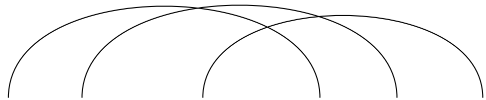
Cardinality of M_n^d and D_n^d (gray) for small values of $d \geq 2, n \geq 0$:

$d \backslash n$	0	1	2	3	4	5	6	*OEIS*
2	1	2	6	20	70	252	924	*A000984*
	1	1	2	5	14	42	132	*A000108*
3	1	6	90	1680	34650	756756	17153136	*A006480*
	1	1	5	42	462	6006	87516	*A005789*
4	1	24	2520	369600	63063000	11732745024	2308743493056	*A008977*
	1	1	14	462	24024	1662804	140229804	*A005790*

Reference MM. A note on multidimensional Dyck languages. In Casadio et al eds, *Categories and Types in Logic, Language, and Physics: Essays Dedicated to Jim Lambek on the Occasion of His 90th Birthday*, pp 279–296. Springer LNCS 8222, 2014.

MCSL: mildly context-sensitive language

Crossing dependencies $a^n b^n c^n$ pattern requiring expressivity beyond context-free



(hij hoopt dat) Jan Marie de kinderen helpt leren fietsen

he hopes that Jan helps Mary to teach the kids how to ride a bike

MCSL (Joshi) family of languages with key properties

- CFL $\subset$ MCSL
- allow bounded degree of crossing dependencies — low bound!
- share polynomial parsability with CFL
- . . .

k-MCFG, k-dimensional context free grammars

MCFG generalize CFG to higher dimensionalities:

- CFG: non-terminals range over strings
- k-MCFG: non-terminals range over string tuples, max size k

Example 2-MCFG for $a^n b^n c^n$

$$\begin{array}{ll} 1. & S(xy) \leftarrow A(x,y) \\ 2. & A(x\,\mathsf{a}, \mathsf{b}\,y\,\mathsf{c}) \leftarrow A(x,y) \\ 3. & A(\epsilon,\epsilon) \leftarrow \end{array}$$

Conjecture (Kanazawa) D^k is recognized by $(k-1)$-MCFG

but . . . coming up with an actual 2-MCFG for D^3 is elusive

Reference Kogkalidis and Melkonian. Towards a 2-Multiple Context-Free grammar for the 3-dimensional Dyck language. In Sikos and Pacuit, eds, *Proceedings ESSLLI Student Session*, LNCS 11667, pp 79–92. Springer, 2018.

D^k and rectangular standard Young tableaux

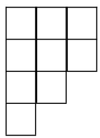
Young tableaux $\rightsquigarrow$ representation theory of the symmetric and general linear groups and . . . Linguistics

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of an integer n, i.e. a multiset of positive integers the sum of which is n, and let us list the k parts in weakly decreasing order.

Diagram arrangement of k boxes into left-aligned rows of length $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$.

Standard Young tableau place the integers 1 through n in the boxes in such a way that the entries are strictly increasing from left to right in the rows and from top to bottom in the columns.

Example $\lambda = (3, 3, 2, 1) \dashv 9$. Diagram (left), standard tableau (right)



1	2	8
3	6	9
4	7	
5		

Yamanouchi words

Given a tableau T of shape $\lambda = (\lambda_1, \ldots, \lambda_k)$, the Yamanouchi word of T is a word $w = w_1 \cdots w_n$ over a k-symbol alphabet $\{1, 2, \ldots, k\}$ such that w_i is the row that contains the integer i in T.

Conversely, given $w \in \{1, 2, \ldots, k\}^+$ with $|w| = n$ and with the property that, reading w from left to right, there are never fewer letters i than letters $(i+1)$, one can recover a tableau with k rows.

Example

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 8 \\ \hline 3 & 6 & 9 \\ \hline 4 & 7 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array} \quad \longleftrightarrow \quad 112342312$$

Rectangular tableaux $\leftrightarrow$ Dyck words

D_n^d words are in bijection with *rectangular* tableaux of shape (d rows $\times$ n columns).

- columns increasing: prefix condition on Dyck words
- rows of same length: equal letter multiplicity

Cardinality Simple form of hook length formula for rectangular diagrams:

$$|D_n^d| \quad = \quad \frac{dn!}{\prod_{k=1}^{n} k^{\overline{d}}}$$

writing $n^{\overline{m}}$ for rising factorial powers $n(n+1)\cdots(n+m-1)$

Example computing $|D_2^3|$

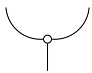
4	3
3	2
2	1

$$\frac{6!}{1 \cdot 2^2 \cdot 3^2 \cdot 4} = \frac{720}{144} = 5$$

Product-coproduct prographs

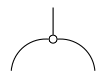
Borie bijection between 3-row, n-column rectangular Young tableaux and $PC(n)$, single-input single-output product coproduct prographs with n product (hence also n coproduct) nodes.

coproduct: single input, two outputs



$$\Delta : V \longrightarrow V \otimes V$$

product: two inputs, single output



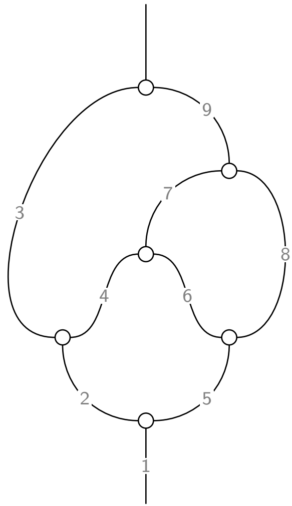
$$\mu : V \otimes V \longrightarrow V$$

- depth-left first traversal
- output edge of a product node can be visited only if its two incoming edges have
- Inputs of coproducts: entries of the top row of the corresponding tableau; left inputs of products: entries of the middle row; right inputs of products of the bottom row

Reference N Borie. Three-dimensional Catalan numbers and product-coproduct prographs. In *FPSAC 2017 The 29th international conference on Formal Power Series and Algebraic Combinatorics*, London. arXiv:1704.00212.

Illustration

Word $aabbacbcc$: prograph with traversal steps, Young tableau



1	2	5
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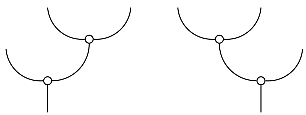
6	8	9

Tamari order for D^3

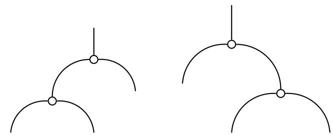
Tamari order, 2D partial ordering on words induced by a semi-associative product operation $(A \bullet B) \bullet C \leq A \bullet (B \bullet C)$

Tamari order, 3D three semi-associative operations on the Borie graphs:

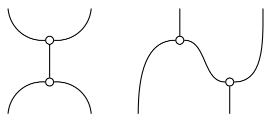
$\rightsquigarrow$ directed versions of Frobenius algebra equations



$\alpha^{\Delta} : (1 \otimes \Delta) \circ \Delta \longrightarrow (\Delta \otimes 1) \circ \Delta$
coproduct semi-associativity

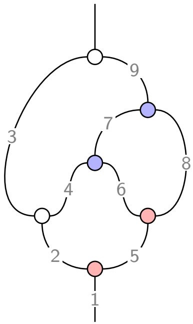


$\alpha_{\mu} : \mu \circ (\mu \otimes 1) \longrightarrow \mu \circ (1 \otimes \mu)$
product semi-associativity



$\alpha_{\mu}^{\Delta} : \Delta \circ \mu \longrightarrow (\mu \otimes 1) \circ (1 \otimes \Delta)$
mixed (co)product semi-associativity

Rewriting $aabbacbcc$



1	2	5
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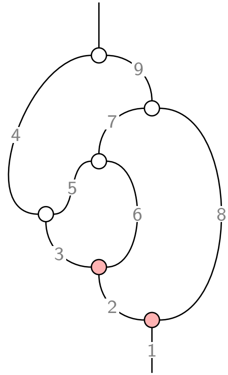

1	2	3
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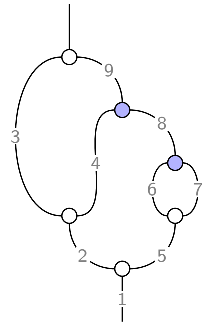
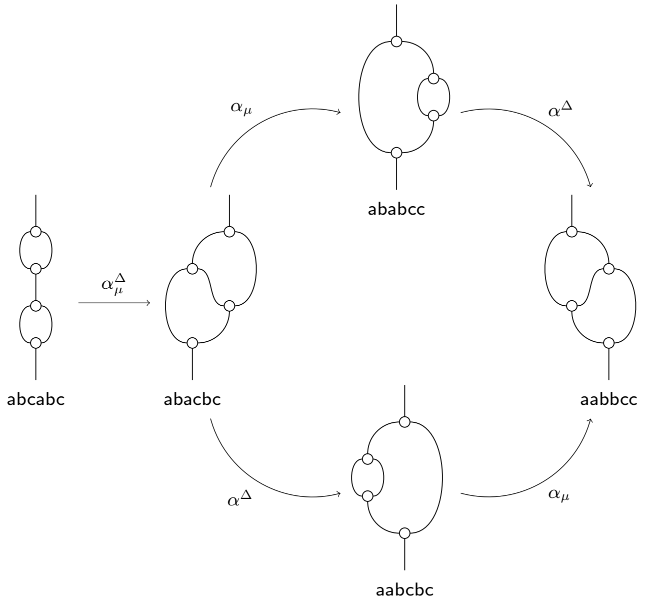


1	2	5
3	4	6
7	8	9

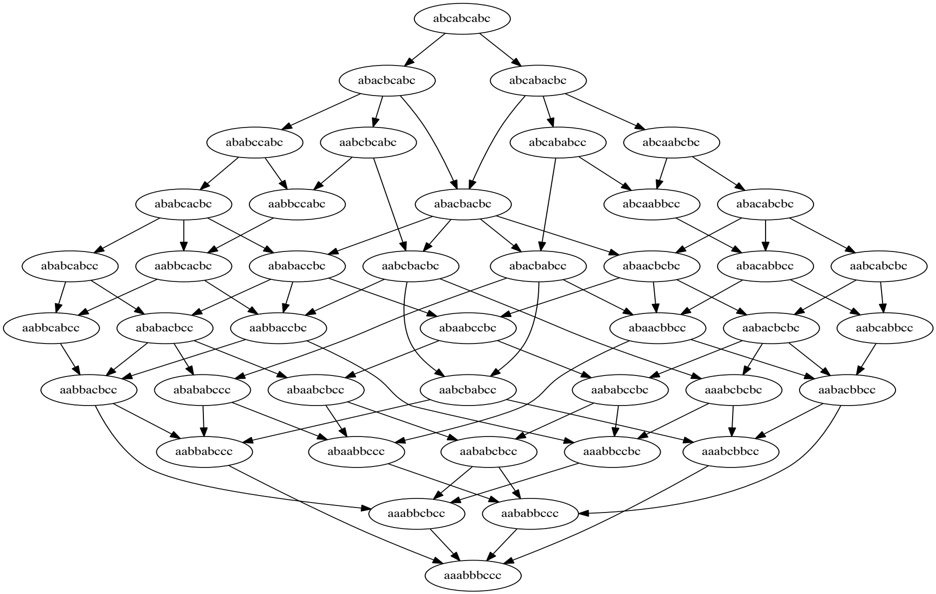
$$aabbacbcc \xrightarrow{\alpha^{\Delta}} aaabbcbcc \quad aabbacbcc \xrightarrow{\alpha_{\mu}} aabbabccc$$

D_2^3 order



Schützenberger duality: $180°$ rotation

D_3^3 order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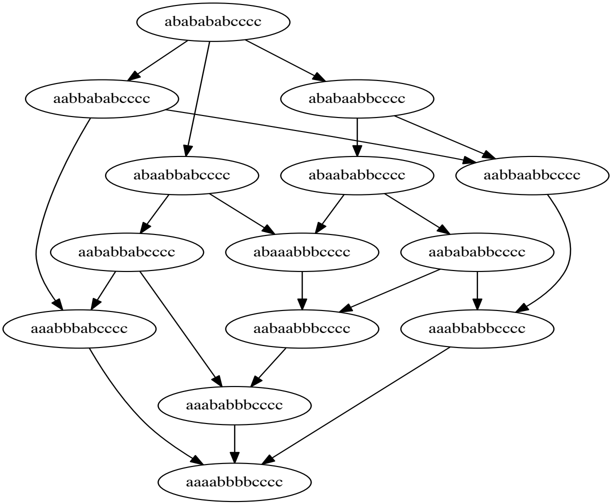
Y_4 subgraph of D_4^3 order



Tamari intervals

Interval $[A, B] = \{C \mid A \leq C \leq B\}$

Counting intervals, 2D rooted 3-connected trivalent maps with $2n+2$ vertices.

$$\frac{2(4n+1)!}{(n+1)!(3n+2)!}$$ (Chapoton 2006)

*OEIS A*000260: 1, 3, 13, 68, 399, . . .

Counting intervals, 3D $1, 14, 453, 22613, 1476916, \ldots$ *OEIS* ??

The Lambek perspective

Zeilberger: $A \leq B$ in 2D Tamari order iff $A \vdash B$ in Lambek's [58] Syntactic Calculus with restricted form of associativity:

$$\frac{A, B, \Gamma \vdash C}{A \bullet B, \Gamma \vdash C} \bullet L' \qquad vs \qquad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \bullet B, \Delta \vdash C} \bullet L$$

Challenge Generalize the approach to D^3, starting from **LG**, Lambek-Grishin calculus extended with semi-associativities?

Refs Zeilberger. A sequent calculus for a semi-associative law. *Log. Methods Comput. Sci.*, 15(1), 2019.

Uustalu, Veltri, and Zeilberger. The sequent calculus of skew monoidal categories. *CoRR*, abs/2003.05213, 2020.

Moortgat and Moot. Proof nets and the categorial flow of information. In Baltag et al, editors, *Logic and Interactive RAtionality. Yearbook 2011*, pages 270–302. 2012.

Lambek-Grishin calculus

Basis Residuated triple $\backslash, \otimes, /$ (product, left/right division) of Lambek's Non-Associative Syntactic Calculus, extended with Grishin's dual residuated triple $\oslash, \oplus, \obslash$ (coproduct, right/left difference)

$$A \longrightarrow C/B \quad \text{iff} \quad A \otimes B \longrightarrow C \quad \text{iff} \quad B \longrightarrow A\backslash C$$

$$B \obslash C \longrightarrow A \quad \text{iff} \quad C \longrightarrow B \oplus A \quad \text{iff} \quad C \oslash A \longrightarrow B$$

Structural extensions same type semi-associativity:

$$(A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \qquad A \oplus (B \oplus C) \longrightarrow (A \oplus B) \oplus C$$

Mixed semi-associativities, aka linear distributivities, Cockett/Seely:

Class I	$A \obslash (B \otimes C) \longrightarrow (A \obslash B) \otimes C$	$(A \otimes B) \oslash C \longrightarrow A \otimes (B \oslash C)$
$\Leftrightarrow$	$(A \oplus B) \otimes C \longrightarrow A \oplus (B \otimes C)$	$A \otimes (B \oplus C) \longrightarrow (A \otimes B) \oplus C$
Class IV	$(A \obslash B) \otimes C \longrightarrow A \obslash (B \otimes C)$	$A \otimes (B \oslash C) \longrightarrow (A \otimes B) \oslash C$

To Do

- find an encoding of a D^3 word as a **LG** formula
- capture Tamari order of D^3 in terms of **LG** derivability
- intuition:
  - same type associativities on pure $\otimes$ and $\oplus$ subformulas
  - mixed formulas with equal number of $\otimes, \oplus$
  - disentangle with the linear distributivities

$\square$

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