Binomial lattice congruences and flat dihomotopy types

Computational Logic and Applications
Kraków

Cameron Calk
& Luigi Santocanale

LABORATOIRE D'INFORMATIQUE & SYSTÈMES

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Introduction

Overview and context

- **Multinomial lattices** were introduced by Bennett & Birkhoff.
  - Study of the rewriting system associated to **commutativity** from a lattice-theoretic perspective:

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- These lattices and their congruences are strongly related to **concurrency**.
  - The word $abbaa$ represents **interleaving actions** of two agents.
  - Multinomial lattice congruences give rise to certain **Parikh equivalences** central to scheduling problems in concurrency.
  - A **geometric interpretation** closely relates these lattices to a semantics of concurrent systems, namely **directed topology**.

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  - Describe the **geometric** intuition of their congruences.

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  - Recall the notion of directed space, and define **cubical complexes**.
  - Introduce the **binomial complex** and describe the dihomotopy types of its subcomplexes.

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- **Result: the correspondence**.
  - Congruences correspond to dihomotopy types of subcomplexes.
  - The congruence lattice of a binomial lattice is isomorphic to the lattice of subcomplex dihomotopy types.

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- We will end by briefly describing **ongoing work** in this area.

Binomial lattices and their congruences

Multinomial lattices

- Given $v \in \mathbb{N}^k$, we denote by $\mathcal{L}(v)$ the set of **words** on the alphabet $\Sigma = \{a_1, \dots, a_k\}$ such that:
  - w ccontains v_i occurrences of the letter a_i.

  We equip this set with the **partial order** generated by

$$w \leq w' \qquad \iff \qquad \exists u, v \quad \begin{cases} w = u \cdot a_i a_j \cdot v \\ w' = u \cdot a_j a_i \cdot v \end{cases} \quad \text{and}\, i < j.$$

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- The poset $(\mathcal{L}(v), \leq)$ has the structure of a **lattice**.

- These structures generalize **permutations** to permutations of multisets, called **multipermutations**.
  - Indeed, for $v = (1, \dots, 1)$, we have $\mathcal{L}(v) = S_k$.
  - The order $\leq$ generalizes the **weak Bruhat order** defining the **permutohedron**.

Binomial lattices

Today, we will focus on **binomial lattices**:

- Given $n, m \in \mathbb{N}$, we denote by $\mathcal{L}(n, m)$ the set of **words** on the alphabet $\Sigma = \{a, b\}$ such that:
  - w contains n occurrences of the letter a,
  - and m occurrences of the letter b.

  which we equip with the **partial order** generated by

$$w \leq w' \qquad \Longleftrightarrow \qquad \exists u, v \quad \begin{cases} w = u \cdot ab \cdot v, \\ w' = u \cdot ba \cdot v. \end{cases}$$

- We will henceforth denote $\mathcal{L}(n, m)$ simply by $\mathcal{L}$.

Proposition (L. Santocanale '05)

*$\mathcal{L}$ is a **distributive** lattice.*

As lattices of lattice paths

- The elements of $\mathcal{L}$ are interpreted as **paths** in an n by m grid:

$$w \in \mathcal{L} \qquad \leftrightsquigarrow \qquad f_w : [n+m] \to [n] \times [m]$$

  - an occurrence of a is a step in the x-axis,
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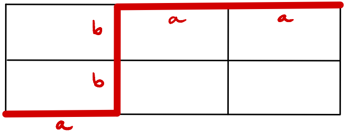
abbaa

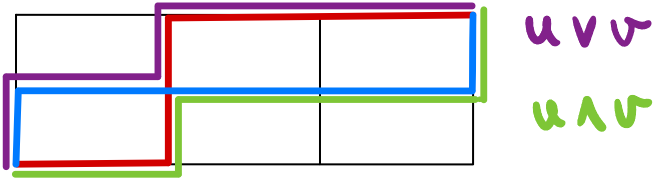


The ordering is recovered as a point-wise ordering on paths.

- The **join** and **meet** relations then become point-wise maxima and minima:

u

v



- Note that these paths are **increasing** in each coordinate.

Distributive lattice congruences

Let L be a distributive lattice.

- A **congruence** on L is an equivalence relation $\theta \subseteq L \times L$ which is compatible with the lattice operations.
- In distributive lattices, congruences are given by **sets** of join-prime elements.
  - $j \in L$ is **join-prime** if

$$j = u \vee v \qquad \Rightarrow \qquad j = u \text{ or } j = v.$$

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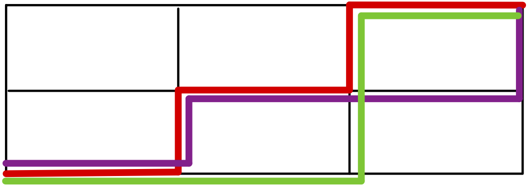
- Given $S \subseteq \mathcal{J}$, the congruence $\equiv_S$ is defined by:

$$u \equiv_S v \qquad \Longleftrightarrow \qquad \forall j \in S, \quad j \leq u \textit{ iff } j \leq v.$$

Join-prime elements of $\mathcal{L}(n, m)$

- What are the join-prime elements of $\mathcal{L}$?



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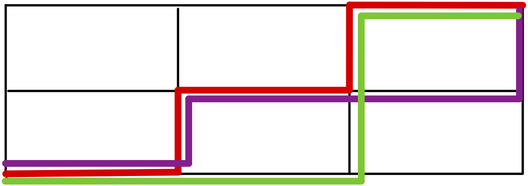

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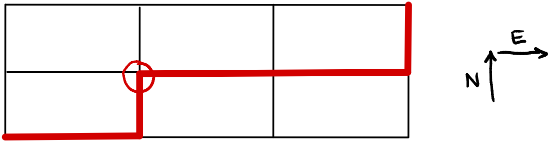
- They are the paths that have exactly one **north-east turn**:



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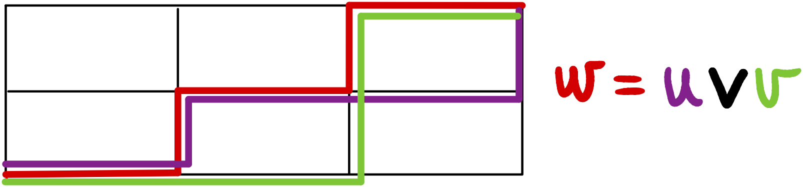

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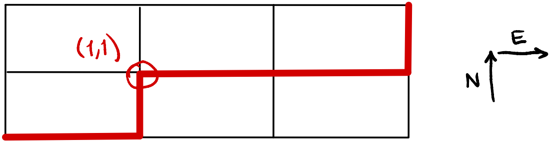

- As **words**, these are of the form

$$a^k b^l a^{n-k} b^{m-l}$$

They are thus characterized by (k,l), with $\begin{cases} 0 \leq k < n \\ 0 < l \leq m \end{cases}$

Geometric interpretation of congruences

- Let us look at the particular case when $S = \{j\}$.
- Recall that

$$w \equiv_S w' \qquad \Longleftrightarrow \qquad j \leq w \;\textit{ iff }\; j \leq w'.$$

- Let (k, l) be the coordinate of the NE turn of j.
  - $j \leq u$ means f_u passes "above" (k, l),
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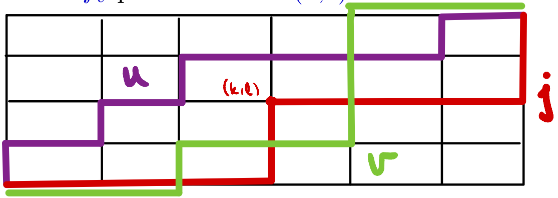

- The same holds for arbitrary $S \subseteq \mathcal{J}$.
- So, lattice congruences of $\mathcal{L}$ correspond to separating directed paths by points. This reminds us of **directed homotopy**...

Directed homotopy and binomial complexes

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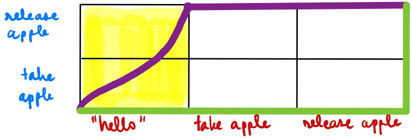
- Today, we focus on a particular class of directed spaces, namely **cubical complexes**. In two dimensions, these consist of:
  - **vertices**, which may be related by…
  - **edges**, which may form the border of…
  - **squares**.
- Such two-dimensional complexes model two-agent concurrent systems:



Bob takes/releases the apple whilst Alice says hello. Then Alice takes/releases the apple.

Alice says hello, takes/releases the apple and then Bob takes/releases the apple.

- Directed paths are those which increase in each coordinate.

Binomial complexes

- In particular, for $n, m \in \mathbb{N}$, we consider the **binomial complex** C:
  - $C_0 := \{v_{(i,j)} \mid 0 \leq i \leq n \text{ and } 0 \leq j \leq m\}$,
  - $C_1 := \{e_{(i_1,j_1),(i_2,j_2)} \mid i_2 = i_1 + 1 \text{ exor } j_2 = j_1 + 1\}$,
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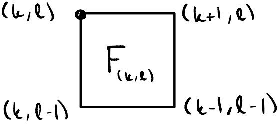
- Note that we encode squares by their **upper-left** corner.



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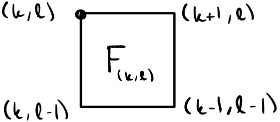
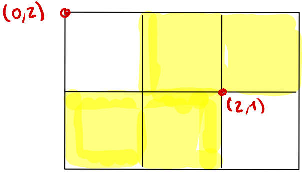


- Given $S \subseteq C_2$, we denote by C^S the cubical complex with the same set of vertices and edges, but in which $C_2^S := C_2 \setminus S$.

C(3,2)

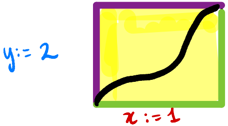


$S = \{(0,2),(2,1)\}$

$C^S(3,2)$

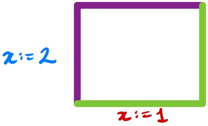
Cubical homotopy

- Given a concurrent system, which executions produce the same output?



All executions end with

1	2
x	y



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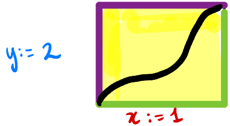

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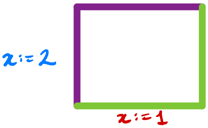

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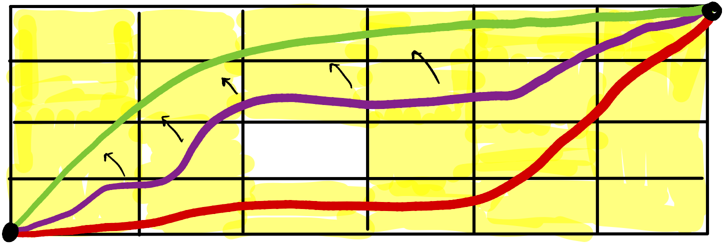

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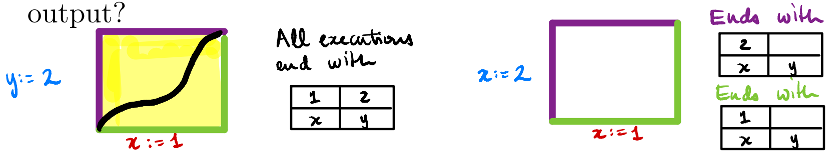

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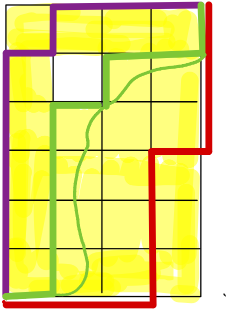
- In a cubical complex Γ, it suffices to consider
  - **combinatorial dipaths**, *i.e.* those which are contained in the set of edges Γ_1,
  - **combinatorial homotopy**, *i.e.* dipaths are equivalent when the space between them is filled by squares in Γ_2.



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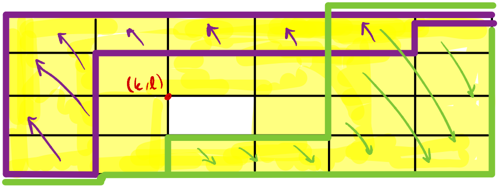
- We are interested in the **quotient** by combinatorial dihomotopy:

$$\overrightarrow{mC\mathbb{P}}(C^S) \Big/ \overset{*}{\leftrightsquigarrow} .$$

- In the particular case in which $S = \{F_{(k,l)}\}$...



Paths going above F are all identified.

Paths going under F are all identified.

The correspondence

Correspondences

- Elements of $\mathcal{L}$.
- Elements of $\overrightarrow{mC\mathbb{P}}(C^S)$.

Lattice paths

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Lattice paths

- Join prime elements of $\mathcal{L}$.
- Squares in C.

$$\mathcal{J} \simeq \{(\mathbf{k}, \mathbf{l}) \mid \mathbf{0} \leq \mathbf{k} < \mathbf{n} \text{ and } \mathbf{0} < \mathbf{l} \leq \mathbf{m}\ \} \simeq \mathbf{C_2}$$

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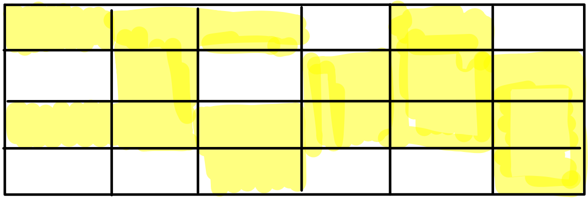
- Congruences $\equiv_S$ of *bilnm*
- Subcomplexes $C^S(n,m)$.



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- Moreover, the maps induced by inclusions $S' \subseteq S$ on each side correspond, *i.e.* the following maps coincide:

$$q_{S',S} : \overrightarrow{mC\mathbb{P}}(C^{S'}) \,/_{\overset{*}{\leftrightsquigarrow}} \longrightarrow \overrightarrow{mC\mathbb{P}}(C^S) \,/_{\overset{*}{\leftrightsquigarrow}}$$
$$p_{S',S} : \mathcal{L}(n,m) \,/_{S'} \longrightarrow \mathcal{L}(n,m) \,/_{S} .$$

Ongoing work

Multinomial lattice quotients

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- In this direction, we are studying the **higher dimensional automata** associated to the multinomial complexes.

The continuous case

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- The continuous analog of the binomial lattice is the **quantale** $Q_\vee(I)$ of sup-continuous endomorphisms of the unit interval I.
  - This is a mix $\star$-autonomous quantale,
  - all permutohedra and binomial lattices are sublattices,
  - elements have interpretation as directed **continuous** paths,
  - congruences are more complicated...

The continuous case

- In what precedes, we have considered **discrete** paths.
- The continuous analog of the binomial lattice is the **quantale** $Q_\vee(I)$ of sup-continuous endomorphisms of the unit interval I.
  - This is a mix $\star$-autonomous quantale,
  - all permutohedra and binomial lattices are sublattices,
  - elements have interpretation as directed **continuous** paths,
  - congruences are more complicated...
- We have related the ordering to directed homotopy, but (it seems) that congruences cannot be captured thereby.
- We are currently exploring other **topological** techniques, *i.e.* Priestley duality, frame duality,... in order to characterize its congruences.
- There are also higher dimensional analogues of this lattice...

Thank you