The combinatorics of free bifibrations

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A functor $p : \mathcal{D} \to \mathcal{C}$ between two categories is a *bifibration* when, roughly speaking, objects of $\mathcal{D}$ may be pushed and pulled along arrows of $\mathcal{C}$. Formally, for any arrow $f : A \to B$ in $\mathcal{C}$ and any object S in $\mathcal{D}$ such that $p(S) = A$, there should be an object $f_* S$ and an arrow $f_S : S \to f_* S$ of $\mathcal{D}$ such that $p(f_S) = f$,

$$\begin{array}{ccc} \mathcal{D} & \qquad & S \overset{f_S}{\dashrightarrow} f_* S \\ {\scriptstyle p}\downarrow & & \\ \mathcal{C} & & A \xrightarrow{f} B \end{array}$$

which are universal in the sense that for any arrow $g : B \to C$ in $\mathcal{C}$ and arrow $\alpha : S \to T$ in $\mathcal{D}$ such that $p(\alpha) = fg$, there is a unique arrow $\beta : f_* S \to T$ such that $\alpha = f_S \beta$.

$$\begin{array}{ccc} S \xrightarrow{\qquad\alpha\qquad} T & & S \xrightarrow{f_S} f_* S \overset{\beta}{\dashrightarrow} T \\ & = & \\ A \xrightarrow{f} B \xrightarrow{g} C & & A \xrightarrow{f} B \xrightarrow{g} C \end{array}$$

Dually, for any arrow $g : B \to C$ in $\mathcal{C}$ and object T in $\mathcal{D}$ such that $p(T) = C$, there should be an object $g^* T$ and an arrow $\bar{g}_T : g^* T \to T$ of $\mathcal{D}$ such that $p(\bar{g}_T) = g$,

$$\begin{array}{c} g^* T \overset{\bar{g}_T}{\dashrightarrow} T \\ \\ B \xrightarrow{g} C \end{array}$$

again universal in the sense that for any arrow $f : A \to B$ in $\mathcal{C}$ and arrow $\alpha : S \to T$ in $\mathcal{D}$ such that $p(\alpha) = fg$, there is a unique arrow $\beta : S \to g^* T$ such that $\alpha = \beta \bar{g}_T$.

$$\begin{array}{ccc} S \xrightarrow{\qquad\alpha\qquad} T & & S \overset{\beta}{\dashrightarrow} g^* T \xrightarrow{\bar{g}_T} T \\ & = & \\ A \xrightarrow{f} B \xrightarrow{g} C & & A \xrightarrow{f} B \xrightarrow{g} C \end{array}$$

An immediate consequence of the definition is that if $p : \mathcal{D} \to \mathcal{C}$ is a bifibration then the operations of pushing or pulling along an arrow $f : A \to B$ of $\mathcal{C}$ extend to a pair of adjoint functors

$$\mathcal{D}_A \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathcal{D}_B \quad (f_* \dashv f^*)$$

where $\mathcal{D}_A$ and $\mathcal{D}_B$ are the *fiber categories* of A and B relative to the functor p, defined as the subcategories of $\mathcal{D}$ spanned by the arrows living over the identities id_A and id_B in $\mathcal{C}$, and indeed any bifibration over $\mathcal{C}$ may be equivalently described by a pseudofunctor $\mathcal{C} \to \mathcal{A}\mathrm{dj}$ into the category of small categories and adjunctions.

The categorical notion of bifibration was originally introduced by Grothendieck, together with the weaker notion of fibration where one only has the ability to pull objects of the category above along arrows of the category below. One reason for the special interest of bifibrations from the perspective of logic and computer science is that the operations of pushing forward or pulling back along an arrow may be seen as generalizations of existential and universal quantification (cf. [MZ16]), and hence by alternating these operations one can in some sense define objects of arbitrary quantifier complexity. The pushforward and pullback operations may also be seen as generalizations of strongest postconditions and weakest preconditions in specification logics.

Although most functors are not bifibrations, any functor $p : \mathcal{D} \to \mathcal{C}$ generates a *free bifibration*, in the sense that there is a bifibration $\tilde{p} : \mathcal{B}\mathrm{Fib}(p) \to \mathcal{C}$ and a functor $\eta_p : \mathcal{D} \to \mathcal{B}\mathrm{Fib}(p)$ such that $p = \tilde{p} \circ \eta_p$. Moreover, the free bifibration is universal in the sense that if $q : \mathcal{E} \to \mathcal{C}$ is any bifibration equipped with a functor $\theta : \mathcal{D} \to \mathcal{E}$ such that $p = q \circ \theta$, then there is an essentially unique morphism of bifibrations $\tilde{\theta} : \mathcal{B}\mathrm{Fib}(p) \to \mathcal{E}$ such that $\theta = \tilde{\theta} \circ \eta_p$. Whereas the free fibration over a functor has a well-known and very simple concrete description, the free bifibration has been relatively little studied, and describing it explicitly is far more subtle. The problem of building the free bifibration over a functor $p : \mathcal{D} \to \mathcal{C}$ is closely related to the problem, studied by Dawson, Paré, and Pronk [DPP03a, DPP03b], of extending $\mathcal{C}$ to a 2-category $\Pi_2\mathcal{C}$ by freely adjoining right adjoints (cf. [SS86]). However, as far as we are aware there is only one direct construction of the free bifibration over a functor in the literature, by Lamarche [Lam10, Lam14],

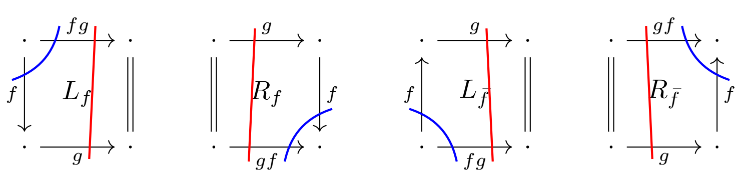
In our work, we have developed a number of alternative descriptions of the free bifibration over an arbitrary functor $p : \mathcal{D} \to \mathcal{C}$. One description is proof-theoretic, viewing the objects of $\mathcal{B}\mathrm{Fib}(p)$ as formulas in a primitive logic containing unary connectives f_* and f^* for every morphism f of $\mathcal{C}$, with the objects of $\mathcal{D}$ serving as atomic formulas. The morphisms of $\mathcal{B}\mathrm{Fib}(p)$ are then defined as equivalence classes of proofs in a simple cut-free sequent calculus containing only four logical rules

$$\frac{S \underset{fg}{\Longrightarrow} T}{f_* S \underset{g}{\Longrightarrow} T}\ L_f \qquad \frac{S \underset{g}{\Longrightarrow} T}{S \underset{gf}{\Longrightarrow} f_* T}\ R_f \qquad \frac{S \underset{g}{\Longrightarrow} T}{f^* S \underset{fg}{\Longrightarrow} T}\ L_{\bar{f}} \qquad \frac{S \underset{gf}{\Longrightarrow} T}{S \underset{g}{\Longrightarrow} f^* T}\ R_{\bar{f}}$$

where proofs are considered modulo four permutation relations, including the relations

$$\frac{\dfrac{S \underset{fh}{\Longrightarrow} T}{S \underset{fhg}{\Longrightarrow} g_* T}\ R_g}{f_* S \underset{hg}{\Longrightarrow} g_* T}\ L_f \sim \frac{\dfrac{S \underset{fh}{\Longrightarrow} T}{f_* S \underset{h}{\Longrightarrow} T}\ L_f}{f_* S \underset{hg}{\Longrightarrow} g_* T}\ R_g \qquad \frac{\dfrac{S \underset{h}{\Longrightarrow} T}{S \underset{hg}{\Longrightarrow} g_* T}\ R_g}{f^* S \underset{fhg}{\Longrightarrow} g_* T}\ L_{\bar{f}} \sim \frac{\dfrac{S \underset{h}{\Longrightarrow} T}{S^* \underset{fh}{\Longrightarrow} T}\ L_{\bar{f}}}{f^* S \underset{fhg}{\Longrightarrow} g_* T}\ R_g$$

as well as their symmetric versions with pushforward and pullback swapped. The cut rule is admissible, thereby defining composition of morphisms in $\mathcal{B}\mathrm{Fib}(p)$. This sequent calculus is closely related to an alternative description of the free bifibration using double category theory. The *double category of zigzags* $\mathbb{Z}\mathcal{C}$ has objects and horizontal arrows given by the objects and arrows of $\mathcal{C}$, vertical arrows given by zigzags (= signed sequences of arrows) in $\mathcal{C}$, and double cells of *zigzag morphisms* generated by vertical pastings of the four generating cells below (ignore the colored arcs for now)



modulo four permutation relations. Composition of zigzag morphisms can be defined inductively by analysis of the intermediate zigzag, thereby defining horizontal composition for the double category. The connection with bifibrations is that $\mathbb{Z}\mathcal{C}$ is the free bifibration over the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, while conversely, any free bifibration may be reconstructed by pulling back the source functor $\mathrm{src} : \mathbb{Z}\mathcal{C} \to \mathcal{C}$ of the double category along an arbitrary functor $p : \mathcal{D} \to \mathcal{C}$. Finally, zigzag morphisms in $\mathbb{Z}\mathcal{C}$ also have a natural graphical interpretation (at least in the case where $\mathcal{C}$ is a free category) as certain planar arc diagrams considered up to isotopy (see colored arcs above).

A challenge in understanding free bifibrations is getting a handle on the equivalence classes of zigzag morphisms induced by the permutation relations. Indeed, by a reduction of [DPP03b], this equivalence relation is in general undecidable! One way we have attacked the problem is via the proof-theoretic technique of *focusing*, developing a (for now conjectured) canonical form whereby permutation equivalence classes of derivations in the above sequent calculus are represented by focused derivations modulo a more elementary notion of "observational" equivalence. In many cases (namely when $\mathcal{C}$ is *factorization preordered*) the observational equivalence relation on focused derivations is just equality, although in general it is undecidable.

Another way we have attacked the problem is by considering examples, and here is where it appears that free bifibrations give rise to a number of categories of great combinatorial interest. A basic example is the free bifibration over the functor $p = (* \mapsto 0) : 1 \to 2$ sending the unique object of 1 to the initial object of the walking arrow category $2 = 0 \to 1$. In this case, objects in the fiber over 0 are isomorphic to even-length sequences of alternating pushes and pulls $f^* f_* \cdots f^* f_* 0$ along the unique arrow $f : 0 \to 1$, while objects in the fiber over 1 correspond to odd-length sequences $f_* f^* f_* \cdots f^* f_* 0$. When we consider morphisms, it turns out that the fiber category $\mathcal{B}\mathrm{Fib}(p)_0$ is equivalent to the (augmented) simplex category Δ of finite ordinals and order-preserving maps, under an interpretation reading the length $2n$ sequence $f^* f_* \cdots f^* f_* 0$ as the ordinal $\mathsf{n} = \{0 < 1 < \cdots < n-1\}$. Similarly, the fiber $\mathcal{B}\mathrm{Fib}(p)_1$ is equivalent to the category $\Delta_\perp$ of finite non-empty ordinals and order-and-least-element-preserving maps. In particular, from the sequent calculus for free bifibrations we can easily derive the well-known formula $\binom{n+m-1}{m}$ for the number of maps $\mathsf{m} \to \mathsf{n}$ in Δ. It is also worth mentioning that in this case the total category of the free bifibration is equivalent $\mathcal{B}\mathrm{Fib}(p) \cong \Upsilon$ to the *category of schedules* Υ introduced by Harmer, Hyland, and Melliès [HHM07] in their study of the categorical combinatorics of game semantics.

An even richer structure emerges considering the free bifibration over the functor $p = (* \mapsto 0) : 1 \to \mathbb{N}$ sending the unique object of 1 to the initial object of the natural numbers considered as a posetal category under the natural order. In this case, objects in the fiber of 0 are isomorphic to sequences of rising and falling steps in $\mathbb{N}$ that start at 0 and end at 0. In other words, they correspond to Dyck paths! By the standard bijection between Dyck paths and rooted planar trees, the fiber $\mathcal{B}\mathrm{Fib}(p)_0$ may therefore be interpreted as a category of trees, giving rise to an interesting notion of *morphism of planar trees*. Indeed, it turns out that $\mathcal{B}\mathrm{Fib}(p)_0$ is equivalent to a category of finite rooted planar trees that was defined in an entirely different manner by Joyal [Joy97] and Batanin [Bat98], namely as the full subcategory of the functor category $[\mathbb{N}^{\mathrm{op}}, \Delta]$ consisting of those functors $T : \mathbb{N}^{\mathrm{op}} \to \Delta$ such that $T(0) = \mathsf{1}$ and such that $T(h) = \mathsf{0}$ for some h. Under the Joyal-Batanin representation of planar trees, the ordinal $T(n)$ counts the number of nodes of height n from the root, while the monotone functions $T(n+1) \to T(n)$ map the nodes of height $n+1$ to their parent nodes of height n (these functions are necessarily order-preserving by planarity). It turns out that natural transformations between such functors are in one-to-one correspondence with equivalence classes of zigzag morphisms between the corresponding Dyck paths. In particular, we can enumerate natural transformations between trees by enumerating focused derivations in the sequent calculus for the free bifibration. Finally, it appears that we get some interesting combinatorics by fixing a tree T and considering the sequences

$$\mathit{in}[T]_n = \#\{\, \alpha : T' \to T \mid |T'| = n \,\} \qquad \mathit{out}[T]_n = \#\{\, \alpha : T \to T' \mid |T'| = n \,\}$$

counting all of the morphisms into T or out of T and out of/into a tree of a given size.

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