“It’s all Greek to me”– on the pre-history of coherence in categorical logic

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This talk details a somewhat whimsical quest to give algebraic / categorical models of ancient Greek logic : precisely, the Stoic treatment of connectives expounded by Chrysippus, as understood – possibly misunderstood – by Hipparchus.

Our starting point is a well-documented disagreement between followers of Hipparchus, and those of Chrysippus, described in Plutarch’s *Quaestiones Conviviales* and other sources. This was on the oddly specific question of how many distinct ‘non-simple assertibles’ [compound propositions] may be formed by conjunctions (or simply ‘combinations’) of ten ‘elementary assertibles’ [atomic propositions]. Chrysippus claimed to give an order of magnitude figure ‘exceeding one million’ (i.e. tens of myriads). In contrast, Plutarch reports — seemingly as a well-established fact — that Hipparchus and ‘all the arithmeticians’ had refuted Chrysippus, and established the precise number to be 103049.

The significance of this number remained unknown until 1994, when it was identified by Daniel Hough as the 10^{th} little Schröder number, and hence the number of rooted planar trees with ten leaves [10]. Taking the very natural step of interpreting each tree leaf as an atomic proposition, and each branching as a connective, we arrive at the ‘obvious interpretation’ of such trees as compound propositions – giving a solution to a very long-standing unresolved question.

The little Schröder numbers are now known as the Schröder-Hipparchus numbers; more generally, it has recently become clear that the combinatorics of ancient Greece was significantly more advanced than previously understood. This, of course, provoked intense interest among historians of science and mathematics, and several convincing attempts (e.g. [1]) have been made to reconstruct how Hipparchus’ calculations could be computed with the framework of ancient Greek mathematics.

Even more recently, the logical system in question has been re-investigated, and found to be similarly ahead of its time. The Stanford Encyclopedia of Philosophy `https://plato.stanford.edu/` describes Chrysipus’ contribution to Greek logic as, “a substructural backwards-working Gentzen-style natural deduction system” – features that are more usually associated with late 20th century logical systems such as [9]. In [3], we find further startlingly modern features such as, *recursively formulated syntax*, *Cut rules*, *Gentzen-style negation introduction*, and *avoidance of paradoxes by deliberate rejection of [structural rules]*.

This leaves us with a puzzle : both Chrysippus and Hipparchus were spectacularly ahead of their time, in logic and combinatorics respectively. How, then, did they arrive at such different figures, and what was the root cause of their disagreement?

This question is analysed by Suzanne Bobzien in [2], who makes the case that, "Hipparchus, as one might expect, got his mathematics right. What I suggest is that he got his Stoic logic wrong". The core of this is that propositions that should have been identified were instead counted separately. Precisely, propositions arising via different (partial or total) bracketings of the same elementary propositions were incorrectly considered as entirely distinct.

This is of course familiar from modern categorical logic; we are often forced to use 'identical up to unique natural isomorphism' instead of some strict notion of equality, and impose coherence conditions that ensure the natural isomorphisms relating distinct bracketings are indeed unique.

A fun question (albeit with no historical justification whatsoever) is to consider how this dispute could be resolved by working within a system where the propositions that Bobzien claims as incorrectly counted separately were indeed related in this way? We give a system that does precisely this. We extend logical models that reflect the known sub-structural properties of Stoic logic[1] to include the assumptions apparently made by Hipparchus.

The result is 'freely generated' in the operadic sense; distinct partial or total bracketings are not identified, & we may therefore label arbitrary facets of Stasheff's associahedra with functors corresponding to partial or total bracketings of conjunctions. We also exhibit natural transformations between them that live in a posetal groupoid, and are therefore unique – thus accounting for coherence & uniqueness in the natural isomorphisms that replace equality.

This gives a range of diagrams, based on associahedra, guaranteed to commute (including, of course, MacLane's pentagon as a very special case). As a rather neat surprise, the components of these natural transformations between 'conjunctions' are the operations used in John Conway's proof[2] of undecidability in elementary arithmetic [4]. Further, they include his motivating example (his claimed 'simplest undecidable arithmetic statement' [5]) as a special case. Finally, we briefly consider some work of John Conway linking this motivating example to some additional classical Greek mathematics.

[1]Precisely, it is believed that the *Weakening* rule was not accepted; some sources also indicate that caution is needed concerning the *Contraction* rule. We therefore borrow and extend a model of conjunction (introduced in [7, 8]) from Multiplicative Linear Logic.

[2]Significantly prefigured by Sergei Maslov [6].

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