Lattices of Paths and Flat Dihomotopy Types

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As described for example in [2, 5], the set of discrete paths in a given grid form a distributive lattice, a binomial lattice, whose underlying ordering is also known as the dominance ordering.

The congruences of this lattice present analogies with notions of homotopy coming from directed topology. In this talk, we formalise this connection by introducing cubical complexes [4], combinatorial directed spaces, whose subspaces' dihomtopy types correspond exactly to lattice quotients. This establishes interesting links between finite lattice theory, combinatorics, and concurrency theory.

This strong connection in the discrete case has led us to extend our investigation to lattices of continuous paths [6]. In this setting, the dominance order is no longer canonically characterised as a point-wise path ordering. We show that such a characterisation can be achieved, similarly as in the discrete case, by constructing simultaneous parametrisations and exhibiting directed homotopies that witness the ordering.

Finally, we shall end this talk by sketching ongoing work relating congruences to (directed) topological properties of the unit square.

1. The discrete case.

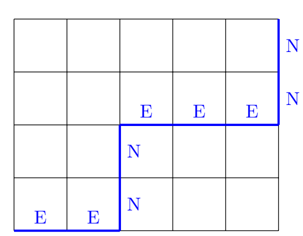
First, we describe the explicit link between the lattice congruences of the binomial lattice $\mathcal{L}(n, m)$ and the dihomotopy types of subspaces of a certain two-dimensional cubical complex $C(n, m)$. Fix non-zero natural numbers $n, m \in \mathbb{N} \setminus \{0\}$.

1.1. Binomial lattice quotients. The binomial lattice, denoted by $\mathcal{L}(n, m)$ or simply $\mathcal{L}$ when the context is clear, admits several equivalent descriptions. Firstly, it is the set of words on the alphabet $\{E, N\}$ having exactly n (resp. m) occurrences of E (resp. N), equipped with the order generated by the relation below on the first line. Secondly, writing $[k] := (0 < \cdots < k)$, $\mathcal{L}$ may be described as the set of monotone (discrete) paths $f : [n + m] \to [n] \times [m]$ on the n by m grid sending 0 to $(0, 0)$ and $n + m$ to (n, m), equipped with the point-wise order defined below on the second line,

$$\begin{aligned} w \to w' \quad &\iff \quad \exists w_1, w_2 \text{ s.t. } w = w_1 E N w_2 \text{ and } w' = w_1 N E w_2 \\ f \leq g \quad &\iff \quad f(k) \leq_2 g(k) \text{ for all } k \in [n + m] \end{aligned}$$

where $\leq_2$ is the product order $[n]^{\mathrm{op}} \times [m]$. These equivalent orderings are known as the *dominance order*. It is well known, see *e.g.* [2] that $\mathcal{L}$ is distributive, and so, being finite, its congruences are given (exactly) by subsets of *join-prime elements*. Here, these are words of the form $E^k N^l E^{\overline{k}} N^{\overline{l}}$, where $(\overline{k}, \overline{l}) = (n, m) - (k, l)$ and $0 \leq x < n$ and $0 < y \leq m$. As paths, these are those having exactly one "north-east" turn, see the figure below.

The join-prime elements of $\mathcal{L}$, the set of which is denoted by J, are therefore characterised by a "point" (k, l) in $[n] \times [m]$ describing the upper-left hand corner of a square in the n by m grid. The congruence $\equiv_S$ associated to a subset $S \subseteq J(\mathcal{L}(n, m))$ identifies $f, g \in \mathcal{L}$ when $j \leq f \iff j \leq g$ for all $j \in S$. This means that $\equiv_S$ identifies all paths which are not separated by the squares associated to the elements of S. This geometric intuition led us to introduce a cubical complex whose subcomplexes' dihomotopical properties correspond to these congruences.



1.2. Binomial cubical complexes. Consider the two-dimensional cubical complex $C(n, m)$ whose set C_0 of vertices is given by elements of $[n] \times [m]$, its set of edges C_1 connect pairs of adjacent vertices, and the set of squares C_2 is given by filling each of the holes in the grid. The latter is thus in bijection with J, since each join-prime uniquely corresponds to a square in the grid. We call $C(n, m)$ the *full* (n, m)*-binomial complex*.

Its geometric realization $||C(n,m)||$ is isomorphic to $[0,n] \times [0,m] \subset \mathbb{R}^2$. In what follows, we will denote this complex simply by C. Given a subset $S \subseteq C_2$, we denote by C^S the cubical complex $(C_0, C_1, C_2 \setminus S)$. Geometrically, $||C^S||$ corresponds to *removing* the squares in S from the space $||C||$.

The *maximal combinatorial directed traces* of C^S, see [3], *i.e.* images of continuous maps $\gamma : \mathbb{I} \to ||C^S||$ such that $\gamma(0) = (0,0)$ and $\gamma(1) = (n,m)$, increasing in each coordinate and which are contained in the 1-skeleton of C^S (the n by m grid), correspond exactly to the elements of $\mathcal{L}$ for any $S \subseteq C_2$. The set of these paths, denoted by $\overrightarrow{\mathfrak{T}}(C^S)$, is equipped with the *elementary dihomotopy relation* [3], denoted $\gamma_1 \rightsquigarrow_S \gamma_2$, which holds when there exist combinatorial dipaths μ and η and a square $F \in C_2 \setminus S$ such that

$$\gamma_1 = \mu \star d_1^1(F) \star d_2^0(F) \star \eta \text{ and } \gamma_2 = \mu \star d_1^0(F) \star d_2^1(F) \star \eta,$$

i.e. γ_1 and γ_2 coincide up to going above or below F. We denote the symmetric, reflexive, transitive closure of $\rightsquigarrow_S$ by $\overset{*}{\leftrightsquigarrow}_S$. Note that the transitive, reflexive closure of $\rightsquigarrow_\emptyset$ corresponds exactly to the ordering $\leq$ on $\mathcal{L}$, thereby naturally equipping $\overrightarrow{\mathfrak{T}}(C)$ with an isomorphic ordering $\preceq$.

Finally, we remark that removing the squares in S from C does not remove any combinatorial dipaths, so we have $\overrightarrow{\mathfrak{T}}(C^S) = \overrightarrow{\mathfrak{T}}(C)$ for all S. We can thus (artificially) equip these sets with the ordering $\preceq$ so that $(\overrightarrow{\mathfrak{T}}(C^S), \preceq) \cong (\mathcal{L}(n,m), \leq)$. The quotients by the dihomotopy relations $\overset{*}{\leftrightsquigarrow}_S$ are compatible with this ordering, so we obtain lattices $\left(\overrightarrow{\mathfrak{T}}(C^S) \big/ \overset{*}{\leftrightsquigarrow}_S, \preceq_S \right)$ for every S.

1.3. Dihomotopy quotients and congruences. Let $j \in J$ be a join-prime element of $\mathcal{L}$ and denote by F_j the corresponding square in C. We establish a technical lemma stating that, given $f \in \mathcal{L}$, we have $j \leq f$ if, and only if, there exists t_f such that the combinatorial trace $\gamma_f : \mathbb{I} \to ||C^S||$ corresponding to f satisfies $\gamma(t_f) \geq_2 (x,y)$ for all $(x,y) \in F_j$. Extending this result to a set of join-primes $S \subseteq J$, we see that $f \equiv_S g$ is equivalent to γ_f and γ_g passing above or below the same set of removed squares in $||C^S||$, *i.e.* having $\gamma_f \overset{*}{\leftrightsquigarrow} \gamma_g$. This leads to our first main result:

Proposition 1. *For any $S \subseteq J$, we have $\mathcal{L}(n,m) \big/ \equiv_S \cong \overrightarrow{\mathfrak{T}}(C^S) \big/ \overset{*}{\leftrightsquigarrow}_S$.*

We can also include the maps induced by inclusions $S \subseteq S'$. Indeed, in this case, $C^{S'}$ is a subcomplex of C^S and we have the inclusion of congruences $\equiv_S \subseteq \equiv_{S'}$. By functoriality, we respectively obtain $q_{S,S'}$: $\overrightarrow{\mathfrak{T}}(C^{S'}) \big/ \overset{*}{\leftrightsquigarrow}_{S'} \to \overrightarrow{\mathfrak{T}}(C^S) \big/ \overset{*}{\leftrightsquigarrow}_S$ and $p_{S,S'} : \mathcal{L}(n,m) \big/ S' \to \mathcal{L}(n,m) \big/ S$. Denoting the associated diagrams by $\overrightarrow{\mathbf{\Pi}}(C(n,m))$ and $\mathbf{Cong}(\mathcal{L}(n,m))$, we have the following:

Theorem 1. $\overrightarrow{\mathbf{\Pi}}(C(n,m)) \cong \mathbf{Cong}(\mathcal{L}(n,m))$.

This means that the congruence lattice $\mathbf{Cong}(\mathcal{L}(n,m))$ of the binomial lattice is exactly the diagram of dihomotopy types of subspaces of the binomial complex. Moreover, each of these is isomorphic to the order dual of the power-set $\mathcal{P}(J)$.

2. Ordering continuous paths.

A third description of the binomial lattice $\mathcal{L}$ is as join-preserving lattice morphisms $f : [n] \to [m]$. A natural continuous extension of these lattices is thus given by studying the lattice of suprema-preserving lattice morphisms of the unit interval $f : \mathbb{I} \to \mathbb{I}$. These may also be seen as *traces*, *i.e.* images of continuous maps $\mathbb{I} \to \mathbb{I}^2$ which are increasing in each coordinate sending 0 to $(0,0)$ and 1 to $(1,1)$. A more complete description of this lattice may be found in [6]. In the discrete case, the ordering on elements of $\mathcal{L}$ are given by the point-wise order induced by $\leq_2$, the product order $[n]^{op} \times [m]$. In the continuous case, the natural ordering on suprema-preserving functions is given by the point-wise order induced by that of $\mathbb{I}$, that is, for such $f, g : \mathbb{I} \to \mathbb{I}$, we set

$$f \leq g \qquad \Longleftrightarrow \qquad f(t) \leq g(t) \text{ for all } t \in \mathbb{I}.$$

Here we briefly describe how this ordering, again called the *dominance order*, is captured the point-wise order over increasing paths $\mathbb{I} \to \mathbb{I}^2$ induced by that of $\mathbb{I}^{op} \times \mathbb{I}$.

2.1. Sup-continuous endomorphisms, traces, and paths. We consider the *quantale of sup-continuous endomorphisms of* $\mathbb{I}$, denoted by $Q_\vee(\mathbb{I})$, *i.e.* the set of suprema-preserving increasing maps $f : \mathbb{I} \to \mathbb{I}$ equipped with the dominance ordering $\leq$ and a *multiplication operation* given by composition of functions.

Each function $f \in Q_\vee(\mathbb{I})$ corresponds uniquely to a *trace*, that is a maximal chain $C_f \subseteq \mathbb{I}^2$. These, in turn, correspond to images of increasing paths $p : \mathbb{I} \to \mathbb{I}^2$ such that $p(0) = (0, 0)$ and $p(1) = (1, 1)$. However, in contrast to the discrete case, in which we essentially parametrize paths by arc-length, many paths correspond to each trace.

For example, the identity endomorphism on $\mathbb{I}$, which corresponds to the diagonal trace Δ, may be parametrized both by $p_1 : t \mapsto (t, t)$ and $p_2 : t \mapsto (t^2, t^2)$. Denoting the product order $\mathbb{I}^{op} \times \mathbb{I}$ by $\leq_2$, clearly we have $p_1 \not\leq_2 p_2$ and $p_2 \not\leq p_1$. Parametrization is thus a major obstruction to describing the dominance order via $\leq_2$.

2.2. Simultaneous parametrization. The previous example illustrates that, for $f, g \in Q_\vee(\mathbb{I})$, we cannot expect $f \leq g$ to imply $p_f \leq_2 p_g$ for any chosen parametrizations p_f and p_g of f and g, respectively. However, note that if f and g are topologically continuous, suitable parametrisations are given by $t \mapsto (t, f(t))\,, (t, g(t))$. In the general case, denoting by D the set of discontinuity of points of f *and* g, we set

$$L = \bigcup_{x \in \mathbb{I}} L_x \qquad \text{where } L_x = \begin{cases} \{x\} \text{ when } x \in \mathbb{I} \setminus D, \\ \mathbb{I} \text{ when } x \in D, \end{cases}$$

This set is equipped with and ordering known as the *order sum* [1] of the family $\{L_x\}$. We show that L is isomorphic to the unit interval $\mathbb{I}$, thereby constructing, for any pair (f, g) of elements of $Q_\vee(\mathbb{I})$, parametrizations π_f and π_g satisfying $\pi_f \leq_2 \pi_g \iff f \leq g$. We call these *simultaneous parametrizations* because their x-coordinates coincide throughout the progression of each path.

Simultaneous parametrisation also relates the ordering on $Q_\vee(\mathbb{I})$ to the notion of directed homotopy. Indeed, consider $f, g \in Q_\vee(\mathbb{I})$ such that $f \leq g$. Then the map $\psi_{f,g} : \mathbb{I} \to [\mathbb{I}, \mathbb{I}^2]$ defined by $\psi_{f,g}(s, t) = (\pi_f^1(t), (1 - s)\pi_f^2(t) + s\pi_g^2(t))$ is an increasing homotopy from π_f to π_g, in the sense that $\psi(s, t) \leq_2 \psi(s', t)$ for all $s \leq s'$. Moreover, any two parametrizations p_f and p'_f of the same element f are related by a homotopy $\varphi_{p,p'}$ with constant image, called a *reparametrization homotopy.* Summing this up, we have

Theorem 2. *Let $f, g \in Q_\vee(\mathbb{I})$ be such that $f \leq g$. Then there exist parametrisations $\pi_f, \pi_g : \mathbb{I} \longrightarrow \mathbb{I}^2$ and a directed homotopy $\psi_{f,g} : \pi_f \longrightarrow \pi_g$. Moreover, given any parametrisations q_f and q_g of f and g, there exist reparametrising homotopies ϕ_f, ϕ_g so that $\phi_f \star \psi_{f,g} \star \phi_g$ is a homotopy from q_f to q_g.*

3. Ongoing work.

To end this talk, we discuss partial results and ongoing work relating congruences of the lattice $Q_\vee(\mathbb{I})$ to topological properties of $\mathbb{I}^2$ and its increasing paths. These include studying the topology of the *Priestley dual space* X of $Q_\vee(\mathbb{I})$ and the subspace X_J correpsonding to its join-primes, as well as its relation to a topology on $\mathbb{I}^2$. We have also observed certain obstructions to generalising Theorem 1 to the continuous case which we will briefly discuss.

References.

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