Counting simply typable lambda terms

Pierre Lescanne

with collaboration of

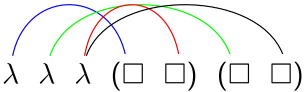
Maciej Bendowsky, Katarzyna Grygiel, Marek Zaionc

Jagiellonian University, ENS de Lyon

CLA, 2016

Bourbaki

- Variables are replaced by boxes.
- Each variables is connected to its binder by a link.



De Bruijn

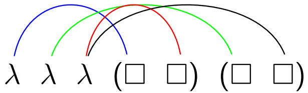
- Links are represented by a natural number.
- The numbers are written using S and $\mathtt{0}$

$$\lambda\lambda\lambda(((SS\mathtt{0})\mathtt{0})((S\mathtt{0})\mathtt{0}))$$

De Bruijn

- Links are represented by a natural number.
- The numbers are written using S and $\mathtt{0}$

$\lambda\lambda\lambda(((SS\mathtt{0})\mathtt{0})((S\mathtt{0})\mathtt{0}))$ represents



Types

If λ-terms represent functions,
types represent *domains* and *co-domains* of λ-terms.

For instance,

- $\lambda\mathtt{0}$: $\alpha \to \alpha$,
- $\lambda\lambda\lambda(((S S \mathtt{0})\mathtt{0})((S\mathtt{0})\mathtt{0}))$: $(\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$

Typing terms

Terms are "typed" in a context Γ, which is a list of types (the types associated with the indices).

Typing terms

Terms are "typed" in a context Γ, which is a list of types (the types associated with the indices).

If $\Gamma = [\alpha_0, \alpha_1, ..., \alpha_n]$ then

- α_0 is the type associated with $\mathtt{0}$,
- α_1 is the type associated with $\mathit{S}\mathtt{0}$,
- etc.

Typing rules

$$\frac{}{\alpha \oplus \Gamma \vdash \mathtt{0} : \alpha} \qquad \frac{\alpha \oplus \Gamma \vdash M : \beta}{\Gamma \vdash \lambda M : \alpha \to \beta}$$

$$\frac{\Gamma \vdash M : \alpha \to \beta \qquad \Gamma \vdash N : \alpha}{M\, N : \beta}$$

Typing rules

$$\frac{}{\alpha \oplus \Gamma \vdash \mathtt{0} : \alpha} \qquad \frac{\alpha \oplus \Gamma \vdash M : \beta}{\Gamma \vdash \lambda M : \alpha \to \beta}$$

$$\frac{\Gamma \vdash M : \alpha \to \beta \qquad \Gamma \vdash N : \alpha}{M\, N : \beta}$$

Definition (Typability)

If one can find a solution to these constraints
for $\Gamma \vdash M : \alpha$, where α is a type and Γ is a context,
the term M is said to be typable in the context M.

Typable closed terms must be typable in the empty context [ ].

Typing rules

$$\frac{}{\alpha \oplus \Gamma \vdash \mathtt{0} : \alpha} \qquad \frac{\alpha \oplus \Gamma \vdash M : \beta}{\Gamma \vdash \lambda M : \alpha \to \beta}$$

$$\frac{\Gamma \vdash M : \alpha \to \beta \qquad \Gamma \vdash N : \alpha}{M\ N : \beta}$$

Definition (Typability)

If one can find a solution to these constraints
for $\Gamma \vdash M : \alpha$, where α is a type and Γ is a context,
the term M is said to be typable in the context M.

Typable closed terms must be typable in the empty context [ ].

Typable closed terms are essentially functional programs.

Examples of typable terms

Typically:

- $\lambda\lambda S\mathtt{0}$ is typable indeed

$$\dfrac{\dfrac{\dfrac{[\alpha] \vdash \mathtt{0} : \alpha}{[\beta, \alpha] \vdash S\mathtt{0} : \alpha}}{[\alpha] \vdash \lambda S\mathtt{0} : \beta \to \alpha}}{[\ ] \vdash \lambda\lambda S\mathtt{0} : \alpha \to \beta \to \alpha}$$

- $\lambda(\mathtt{0}\ \mathtt{0})$ is not typable indeed

$$\dfrac{\dfrac{[\alpha] \vdash \mathtt{0} : \alpha \to \beta \qquad [\alpha] \vdash \mathtt{0} : \alpha}{[\alpha] \to \mathtt{0}\ \mathtt{0} : \beta}}{[\ ] \vdash \lambda(\mathtt{0}\ \mathtt{0}) : \alpha \to \beta}$$

Examples of typable terms

Typically:

- $\lambda\lambda S\mathtt{0}$ is typable indeed

$$\dfrac{\dfrac{\dfrac{[\alpha] \vdash \mathtt{0} : \alpha}{[\beta, \alpha] \vdash S\mathtt{0} : \alpha}}{[\alpha] \vdash \lambda S\mathtt{0} : \beta \to \alpha}}{[\,] \vdash \lambda\lambda S\mathtt{0} : \alpha \to \beta \to \alpha}$$

- $\lambda(\mathtt{0}\ \mathtt{0})$ is not typable indeed

$$\dfrac{\dfrac{[\alpha] \vdash \mathtt{0} : \alpha \to \beta \qquad [\alpha] \vdash \mathtt{0} : \alpha}{[\alpha] \to \mathtt{0}\ \mathtt{0} : \beta}}{[\,] \vdash \lambda(\mathtt{0}\ \mathtt{0}) : \alpha \to \beta}$$

Stability by encompassment

If a term contains a non typable subterm, it is not typable.

If a term contains $\lambda(\mathtt{0}\,\mathtt{0})$ it is not typable,

Stability by encompassment

If a term contains a non typable subterm, it is not typable.

If a term contains $\lambda(\mathtt{0}\,\mathtt{0})$ it is not typable,

as are terms containing
$\lambda(\mathtt{00}), \lambda(\lambda(\mathtt{00})), \lambda(\lambda(\lambda(\mathtt{00}))),$
$\lambda(\mathtt{0}\lambda(\mathtt{1})), \lambda(\mathtt{0}(\mathtt{00})), \lambda((\mathtt{00})\mathtt{0}),$
$\lambda(\lambda(\lambda(\lambda(\mathtt{00})))), \lambda(\lambda(\mathtt{0}\lambda(\mathtt{1}))), \lambda(\lambda(\mathtt{0}(\mathtt{00}))),$
$\lambda(\lambda(\mathtt{11})), \lambda(\lambda((\mathtt{00})\mathtt{0})), \lambda(\mathtt{0}\lambda(\mathtt{00})),$
$\lambda(\mathtt{0}(\lambda(\mathtt{0})\mathtt{0})), \lambda(\lambda(\mathtt{0})(\mathtt{00})), \lambda((\mathtt{00})\lambda(\mathtt{0})),$
$\lambda(\lambda(\mathtt{00})\mathtt{0}), \lambda((\mathtt{0}\lambda(\mathtt{0}))\mathtt{0}), \lambda((\lambda(\mathtt{0})\mathtt{0})\mathtt{0}),$
$(\lambda(\mathtt{0})\lambda(\mathtt{00})), (\lambda(\mathtt{00})\lambda(\mathtt{0}))$

Stability by encompassment

If a term contains a non typable subterm, it is not typable.

If a term contains $\lambda(\mathtt{0}\ \mathtt{0})$ it is not typable,

as are terms containing
$\lambda(\mathtt{00}), \lambda(\lambda(\mathtt{00})), \lambda(\lambda(\lambda(\mathtt{00}))),$
$\lambda(\mathtt{0}\lambda(\mathtt{1})), \lambda(\mathtt{0}(\mathtt{00})), \lambda((\mathtt{00})\mathtt{0}),$
$\lambda(\lambda(\lambda(\lambda(\mathtt{00})))), \lambda(\lambda(\mathtt{0}\lambda(\mathtt{1}))), \lambda(\lambda(\mathtt{0}(\mathtt{00}))),$
$\lambda(\lambda(\mathtt{11})), \lambda(\lambda((\mathtt{00})\mathtt{0})), \lambda(\mathtt{0}\lambda(\mathtt{00})),$
$\lambda(\mathtt{0}(\lambda(\mathtt{0})\mathtt{0})), \lambda(\lambda(\mathtt{0})(\mathtt{00})), \lambda((\mathtt{00})\lambda(\mathtt{0})),$
$\lambda(\lambda(\mathtt{00})\mathtt{0}), \lambda((\mathtt{0}\lambda(\mathtt{0}))\mathtt{0}), \lambda((\lambda(\mathtt{0})\mathtt{0})\mathtt{0}),$
$(\lambda(\mathtt{0})\lambda(\mathtt{00})), (\lambda(\mathtt{00})\lambda(\mathtt{0}))$

or simply terms
$\mathtt{0}\ \mathtt{0}, \mathtt{0}\ \lambda(\mathtt{1}), \mathtt{1}\ \mathtt{1}, \lambda(\mathtt{0}(\lambda(\mathtt{0})\mathtt{0})), \lambda((\lambda(\mathtt{0})\mathtt{0})\mathtt{0})$

How to measure links?

There are at least four possibilities to measure the size of links:

1. Links have no size,

How to measure links?

There are at least four possibilities to measure the size of links:

1. Links have no size,
2. Links have size 1,

How to measure links?

There are at least four possibilities to measure the size of links:

1. Links have no size,
2. Links have size 1,
3. Links have size their length,

How to measure links?

There are at least four possibilities to measure the size of links:

1. Links have no size,
2. Links have size 1,
3. Links have size their length,
4. Terms have size their binary representation (John Tromp).

Operators have size 1 and indices have size 0

$$
\begin{aligned}
|\lambda M| &= |M| + 1 \\
|M_1\, M_2| &= |M_1| + |M_2| + 1 \\
|S^k n| &= 0
\end{aligned}
$$

Operators have size 1 and indices have size 0

$$
\begin{array}{rcl}
|\lambda M| & = & |M| + 1 \\
|M_1\ M_2| & = & |M_1| + |M_2| + 1 \\
|S^k n| & = & 0
\end{array}
$$

The growth of the number of terms of size n is hyper-exponential.

Operator size 1, index size 0: **Ranking and Unranking**

- One can rank the terms of size n from 1 to S_n.
- Symmetrically one can unrank the terms, i.e., from a number j in $[1..S_n]$ one can produce the term having rank j.
- Therefore, one can list the terms of size n and test their typability.
- Hence one can list the typable terms.
- And one can count typable terms of size n.
- One can draw a random number in $[1..S_n]$, unrank it and check whether it is typable.
- One can uniformly generate a random typable terms.

Operator size 1, index size 0: **Ratio of typables**

size	1	2	3	4	5	6	7
nb of terms	1	3	14	82	579	4 741	43 977
nb of typables	1	2	9	40	238	1 564	11 807
ratio	1	0.666	0.6428	0.4878	0.4110	0.3299	0.2684

size	8	9	10	11	12
nb of terms	454 283	5 159 441	63 782 411	851 368 766	12 188 927 818
nb of typables	98 529	904 318	9 006 364	96 709 332	1 110 858 977[1]
ratio	0.2168	0.1752	0.1412	0.1136	0.0911

[1] Paul Tarau

Operator size 1, index size 0: **Ratio of typables**

size	1	2	3	4	5	6	7
nb of terms	1	3	14	82	579	4 741	43 977
nb of typables	1	2	9	40	238	1 564	11 807
ratio	1	0.666	0.6428	0.4878	0.4110	0.3299	0.2684

size	8	9	10	11	12
nb of terms	454 283	5 159 441	63 782 411	851 368 766	12 188 927 818
nb of typables	98 529	904 318	9 006 364	96 709 332	1 110 858 977[1]
ratio	0.2168	0.1752	0.1412	0.1136	0.0911

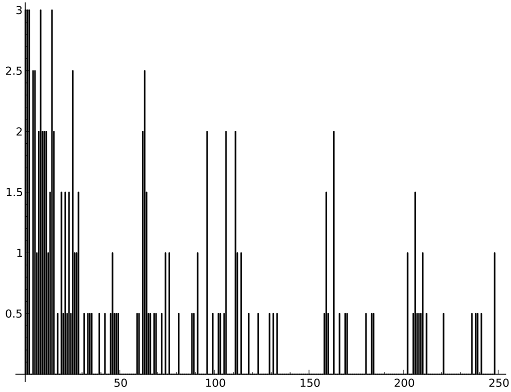
Further by Monte Carlo method:

size	13	14	15	16	20	30	40	45	50
ratio	.073	.056	.047	.039	.0014	.0012	.0003	.00005	$<10^{-5}$

[1] Paul Tarau

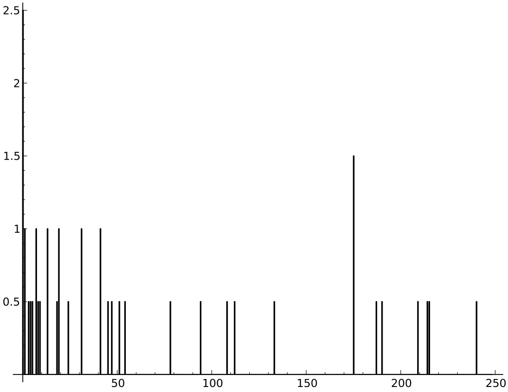
Operator size 1, index size 0: **Distribution of typables**

This method for generating random typable terms of a given size is inefficient, since typable terms are rare.



Distribution of simply typed lambda terms of size 25.
250 segments on the horizontal axis, percentage of typable terms in segments on the vertical axis.

Operator size 1, index size 0: **Distribution of typables**



Distribution of simply typed lambda terms of size 30.
250 segments on the horizontal axis, percentage of typable terms in segments on the vertical axis.

Tromp size

$$
\begin{aligned}
|\lambda M| &= |M| + 2 \\
|M_1\ M_2| &= |M_1| + |M_2| + 2 \\
|Sn| &= |n| + 1 \\
|\mathtt{0}| &= 1.
\end{aligned}
$$

Tromp size

Theorem

The number of all binary λ-terms of size n satisfies

$$S_{\infty,n} \sim (1/\rho)^n \cdot \frac{C}{n^{3/2}},$$

where $\rho \doteq 0.509308127$ and $C \doteq 1.021874073$.

Tromp size

Theorem

The number of all binary λ-terms of size n satisfies

$$S_{\infty,n} \sim (1/\rho)^n \cdot \frac{C}{n^{3/2}},$$

where $\rho \doteq 0.509308127$ *and* $C \doteq 1.021874073$.

Theorem (Bernhard Gittenberger and Zbigniew Gobiewski)

For closed terms of size n; there exist two constants $\underline{C}$ *and* $\overline{C}$ *such that*

$$\liminf \frac{S_{0,n}}{\rho^{-n}\underline{C}n^{-3/2}} \geq 1 \quad \textit{and} \quad \limsup \frac{S_{0,n}}{\rho^{-n}\overline{C}n^{-3/2}} \leq 1$$

Numbers of typable terms for Tromp size

n	$S_{0,n}$
0	0
1	0
2	0
3	0
4	1
5	0
6	1
7	1
8	2
9	1
10	6
11	5
12	13
13	14
14	37
15	44
16	101
17	134
18	298
19	431
20	883
21	1361
22	2736
23	4405
24	8574
25	14334
26	27465
27	47146
28	89270
29	156360
30	293840
31	522913
32	978447
33	1761907
34	3288605
35	5977863
36	11148652
37	20414058
38	38071898
39	70125402
40	130880047
41	242222714
42	452574468
43	840914719
44	1573331752
45	2 933 097 201
46	5 495 929 096

n	$T_{0,n}$
0	0
1	0
2	0
3	0
4	1
5	0
6	1
7	1
8	1
9	1
10	5
11	4
12	9
13	13
14	23
15	29
16	67
17	94
18	179
19	285
20	503
21	795
22	1503
23	2469
24	4457
25	7624
26	13475
27	23027
28	41437
29	72165
30	128905
31	227510
32	405301
33	715078
34	1280127
35	2279393
36	4086591
37	7316698
38	13139958
39	23551957
40	42383667
41	76278547
42	137609116
43	248447221
44	449201368
45	812 315 229
46	1 470 997 501

n	$S_{\infty,n}$
0	0
1	0
2	1
3	1
4	2
5	2
6	4
7	5
8	10
9	14
10	27
11	41
12	78
13	126
14	237
15	399
16	745
17	1292
18	2404
19	4259
20	7915
21	14242
22	26477
23	48197
24	89721
25	164766
26	307294
27	568191
28	1061969
29	1974266
30	3698247
31	6905523
32	12964449
33	24295796
34	45711211
35	85926575
36	161996298
37	305314162
38	576707409
39	1089395667
40	2061428697
41	3 901 829 718
42	7 395 529 009
43	14 023 075 765
44	26 620 080 576
45	50 556 677 634
46	96 108 150 292

n	$T_{\infty,n}$
0	0
1	0
2	1
3	1
4	2
5	2
6	3
7	5
8	8
9	13
10	22
11	36
12	58
13	103
14	177
15	307
16	535
17	949
18	1645
19	2936
20	5207
21	9330
22	16613
23	29921
24	53588
25	96808
26	174443
27	316267
28	572092
29	1040596
30	1888505
31	3441755
32	6268500
33	11449522
34	20902152
35	38256759
36	70004696
37	128336318
38	235302612
39	432050796
40	793513690
41	1 459 062 947
42	2 683 714 350
43	*unknown*
44	*unknown*
45	*unknown*
46	*unknown*

Everything has size 1

A natural counting is to assign size 1 to each operator:

$$
\begin{array}{rcl}
|\lambda M| & = & |M| + 1 \\
|M_1\ M_2| & = & |M_1| + |M_2| + 1 \\
|Sn| & = & |n| + 1 \\
|\mathtt{0}| & = & 1.
\end{array}
$$

Everything has size 1

A natural counting is to assign size 1 to each operator:

$$
\begin{aligned}
|\lambda M| &= |M| + 1 \\
|M_1\, M_2| &= |M_1| + |M_2| + 1 \\
|Sn| &= |n| + 1 \\
|\mathtt{0}| &= 1.
\end{aligned}
$$

$$|\lambda\lambda\lambda(((SS\mathtt{0})\mathtt{0})((S\mathtt{0})\mathtt{0}))| = 13.$$

The sequence **A105633**

The 18 first values of $[z^n]L_\infty$ are:

0, 1, 2, 4, 9, 22, 57, 154, 429, 1223, 3550, 10455, 31160, 93802, 284789, 871008, 2681019

which is the sequence **A105633**.

How about typed terms?

size	typables	all	ratio
1	0	0	
2	1	1	1
3	1	1	1
4	2	3	0.6666
5	5	6	0.8333
6	13	17	0.7647
7	27	41	0.6585
8	74	116	0.6379
9	198	313	0.6325
10	508	895	0.5675
11	1 371	2 550	0.5376
12	3 809	7 450	0.5221
13	10 477	21 881	0.4788
14	29 116	65 168	0.4467
15	82 419	195 370	0.4219
16	233 748	591 007	0.3955
17	666 201	1 798 718	0.3704
18	1 914 668	5 510 023	0.3474
19	5 528 622	16 966 529	0.3259
20	16 019 330	52 506 837	0.3051
21	46 642 245	163 200 904	0.2858
22	136 326 126	509 323 732	0.2677
23	399 652 720	1 595 311 747	0.2505
24	1 175 422 931	5 013 746 254	0.2344

Natural size

size	typables	all	ratio
1	0	0	
2	1	1	1
3	1	1	1
4	2	3	0.6666
5	5	6	0.8333
6	13	17	0.7647
7	27	41	0.6585
8	74	116	0.6379
9	198	313	0.6325
10	508	895	0.5675
11	1 371	2 550	0.5376
12	3 809	7 450	0.5221
13	10 477	21 881	0.4788
14	29 116	65 168	0.4467
15	82 419	195 370	0.4219
16	233 748	591 007	0.3955
17	666 201	1 798 718	0.3704
18	1 914 668	5 510 023	0.3474
19	5 528 622	16 966 529	0.3259
20	16 019 330	52 506 837	0.3051
21	46 642 245	163 200 904	0.2858
22	136 326 126	509 323 732	0.2677
23	399 652 720	1 595 311 747	0.2505
24	1 175 422 931	5 013 746 254	0.2344

Size 0
for variables

size	2	3	4	5	6	7
terms	3	14	82	579	4 741	43 977
typables	2	9	40	238	1 564	11 807
ratio	0.666	0.6428	0.4878	0.4110	0.3299	0.2684

size	8	9	10	11	12
terms	454 283	5 159 441	63 782 411	851 368 766	12 188 927 818
typables	98 529	904 318	9 006 364	96 709 332	1 110 858 977
ratio	0.2168	0.1752	0.1412	0.1136	0.0911

Generation of closed typable terms

- One can speed up the generating process, for instance using Boltzmann samplers, and then sieving.
  But this is still inefficient, since there are too few typable terms.

Generation of closed typable terms

- One can speed up the generating process, for instance using Boltzmann samplers, and then sieving.
  But this is still inefficient, since there are too few typable terms.
- How to generate random typable terms directly?

Conclusion

Counting and generating typable terms is a challenge!

Conclusion

Counting and generating typable terms is a challenge!

even simply typable terms!