A natural counting of lambda terms

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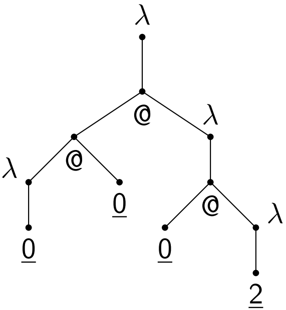
Workshop on Computational Logic and Applications
Novi Sad, May 2016

Motivations

- Combinatorics – design new methods for counting structures with binders and local scopes
- Computer Science – develop tools for random λ-term generation used in software testing (see, e.g. Quickcheck)
- Computational Logic – study quantitative aspects of semantic properties in λ-calculus and related systems
- ...

Natural size notion



$$\lambda x.((\lambda y.y)x)(\lambda z.z(\lambda w.x))$$

$$\lambda((\lambda\underline{0})\underline{0})(\lambda\underline{0}(\lambda\underline{2}))$$

$$|t| = 13$$

Natural counting of λ-terms

$\mathcal{D}$ — de Bruijn indices
$\mathcal{L}_\infty$ — all λ-terms (not necessarily closed)

$$\begin{aligned} \mathcal{D} &= \mathtt{0} \oplus S\,\mathcal{D} \\ \mathcal{L}_\infty &= \mathcal{D} \oplus \mathcal{L}_\infty \mathcal{L}_\infty \oplus \lambda \mathcal{L}_\infty \end{aligned}$$

$$\downarrow$$

$$\begin{aligned} L_\infty(z) &= zL_\infty^2(z) + zL_\infty(z) + \frac{z}{1-z} \\ L_\infty(z) &= \frac{(1-z)^{3/2} - \sqrt{1-3z-z^2-z^3}}{2z\sqrt{1-z}} \end{aligned}$$

The number of λ-terms

Theorem

The asymptotic approximation of the number of λ-terms of size n is given by

$$[z^n]L_\infty(z) \sim (3.38298\ldots)^n \frac{C}{n^{3/2}}, \quad \text{where} \quad C = 0.60676\ldots$$

Holonomic presentation of L_∞

$$L_\infty(z) = \frac{(1-z)^{3/2} - \sqrt{1-3z-z^2-z^3}}{2z\sqrt{1-z}}$$

$$\downarrow$$

Maple: package gfun

$$z^3 + z^2 - 2z + (z^3 + 3z^2 - 3z + 1)L_\infty + (z^5 + 2z^3 - 4z^2 + z)L'_\infty = 0$$

Computing $L_{\infty,n}$

Linear recursion for $L_{\infty,n}$

$$L_{\infty,0} = 0, \qquad L_{\infty,1} = 1, \qquad L_{\infty,2} = 2, \qquad L_{\infty,3} = 4,$$

$$\begin{aligned}(n+1)L_{\infty,n} &= (4n-1)L_{\infty,n-1} - (2n-1)L_{\infty,n-2} \\ &\quad - L_{\infty,n-3} - (n-4)L_{\infty,n-4}\end{aligned}$$

λ-terms with bounded number of free indices

$$\begin{array}{rcl}\mathcal{D}_m & = & \{\underline{0}, \underline{1}, \ldots, \underline{\text{m-1}}\} \\ \mathcal{L}_m & = & \mathcal{D}_m \oplus \mathcal{L}_m \, \mathcal{L}_m \oplus \lambda \, \mathcal{L}_{m+1}\end{array}$$

$$\downarrow$$

$$L_m(z) = \frac{1 - \sqrt{1 - 4z^2 \left(L_{m+1}(z) + \frac{1-z^m}{1-z} \right)}}{2z}$$

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L_m is expressed by means of infinitely nested radicals!

Counting λ-terms containing fixed subterms

Theorem

For a fixed term M, the asymptotic density of $\mathcal{T}_{\mathsf{M}}$ is equal to 1. In other words, asymptotically almost all λ-terms contain M as a subterm.

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Proof sketch

1. $\mathcal{T}_{\mathsf{M}} = \mathsf{M} \oplus \lambda\,\mathcal{T}_{\mathsf{M}} \oplus \mathcal{T}_{\mathsf{M}}\,\mathcal{L}_\infty \oplus \mathcal{L}_\infty\,\mathcal{T}_{\mathsf{M}} \ominus \mathcal{T}_{\mathsf{M}}\,\mathcal{T}_{\mathsf{M}}$.
2. Consider $L_\infty(z) - T_{\mathsf{M}}(z)$. Show that it has density 0.

Counting λ-terms containing fixed subterms (cont'd)

Theorem

Asymptotically almost no λ-term is strongly normalizing.

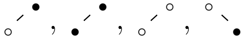
The sequence enumerating λ-terms

The sequence $([z^n]L_\infty(z))_{n\in\mathbb{N}}$ is known as **A105633** in the *Online Encyclopedia of Integer Sequences* (`http://oeis.org`)!

0, 1, 2, 4, 9, 22, 57, 154, 429, 1223, 3550, 10455, 31160, 93802, 284789, 871008, 2681019, 8298933, 25817396,...

E-free black-white binary trees

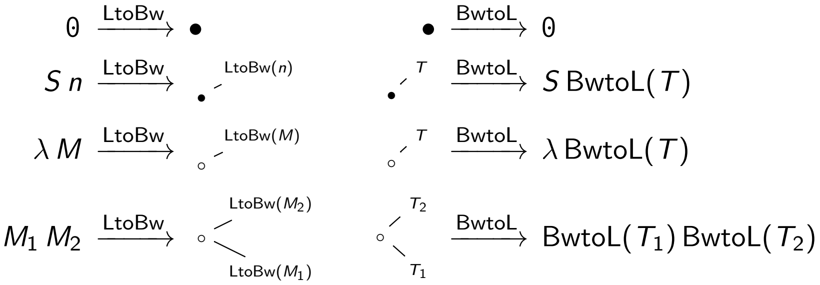
Black-white binary trees (A105633)

1. $A = \{$  $\}$,
2. Roots are black.

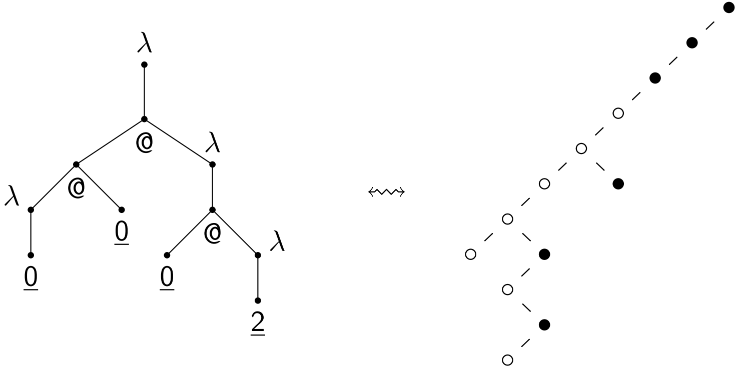
$\downarrow$

$$
\begin{aligned}
BW_{\bullet}(z) &= z + zBW_{\bullet}(z) + zBW_{\circ}(z) \\
BW_{\circ}(z) &= z + zBW_{\circ}(z) + zBW_{\bullet}(z) + zBW_{\circ}(z)BW_{\bullet}(z)
\end{aligned}
$$

Black-white trees and λ-terms: back and forth

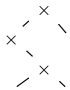


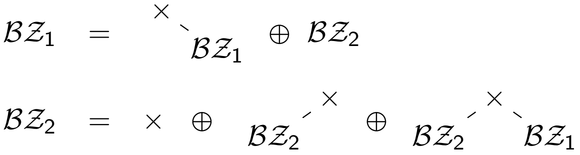
Example



Binary trees without zigzags

Zigzag-free binary trees (**A105633**)

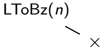




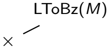
From λ-terms to zigzag-free trees

$\mathsf{LToBz}(\underline{0}) = \times$

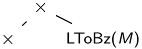
$\mathsf{LToBz}(S(n)) =$



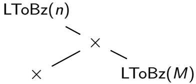
$\mathsf{LToBz}(\lambda M) =$



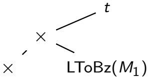
$\mathsf{LToBz}(M\,\underline{0}) =$



$\mathsf{LToBz}(M\,S(n)) =$



$\mathsf{LToBz}(M_1\,M_2) =$



when $\mathsf{LToBz}(M_2) =$

From zigzag-free trees to λ-terms

$$\mathrm{BzToL}(\times) = \underline{0}$$

$$\mathrm{BzToL}\left(\begin{array}{ccc} n & & \\ & \diagdown & \\ & & \times \end{array}\right) = S(\mathrm{BzToL}\,(n))$$

$$\mathrm{BzToL}\left(\begin{array}{ccc} & & \times \\ & \diagup & \\ \times & & \end{array}\right) = \lambda \underline{0}$$

$$\mathrm{BzToL}\left(\begin{array}{ccccc} & & \times & & \\ & \diagup & & \diagdown & \\ \times & & & & T \end{array}\right) = \mathrm{BzToL}(T)\,\underline{0}$$

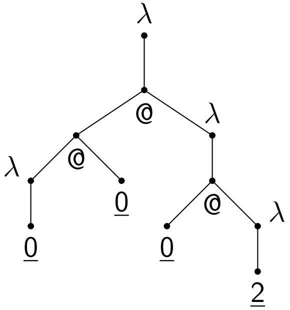
$$\mathrm{BzToL}\left(\begin{array}{ccc} n & & \\ & \diagdown & \\ & & \times \\ & \diagup & \\ \times & & \end{array}\right) = \lambda\,\mathrm{BzToL}\left(\begin{array}{ccc} n & & \\ & \diagdown & \\ & & \times \end{array}\right)$$

$$\mathrm{BzToL}\left(\begin{array}{ccccc} n & & & & \\ & \diagdown & & & \\ & & \times & & \\ & \diagup & & \diagdown & \\ \times & & & & T \end{array}\right) = \mathrm{BzToL}(T)\ \lambda\,\mathrm{BzToL}\left(\begin{array}{ccc} n & & \\ & \diagdown & \\ & & \times \end{array}\right)$$

$$\mathrm{BzToL}\left(\begin{array}{ccccc} & & & & T \\ & & & \diagup & \\ & & \times & & \\ & \diagup & & & \\ \times & & & & \end{array}\right) = \lambda\,\mathrm{BzToL}\left(\begin{array}{ccc} & & T \\ & \diagup & \\ \times & & \end{array}\right)$$

$$\mathrm{BzToL}\left(\begin{array}{ccccc} & & & & T_2 \\ & & & \diagup & \\ & & \times & & \\ & \diagup & & \diagdown & \\ \times & & & & T_1 \end{array}\right) = \mathrm{BzToL}(T_1)\ \mathrm{BzToL}\left(\begin{array}{ccc} & & T_2 \\ & \diagup & \\ \times & & \end{array}\right)$$

Example



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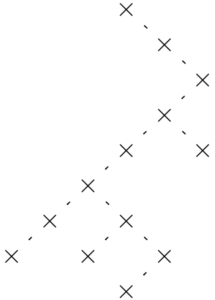

Normal forms

$\mathcal{N}$ — normal forms (λ-terms without redexes)
$\mathcal{M}$ — neutral terms (normal forms without head abstractions)

$$\begin{aligned} \mathcal{N} &= \mathcal{M} + \lambda\mathcal{N} \\ \mathcal{M} &= \mathcal{M}\mathcal{N} + \mathcal{D} \\ \mathcal{D} &= S\mathcal{D} + \mathtt{0} \end{aligned}$$

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$$\downarrow$$

$$\begin{aligned} M(z) &= \frac{1 - z - \sqrt{(1+z)(1-3z)}}{2z} \\ N(z) &= \frac{M(z)}{1-z} \end{aligned}$$

Neutral terms and Motzkin trees

u_n — chain with n vertices
λ_n — chain with n abstractions

$$\mathsf{UnToL}(\bullet) = \lambda$$

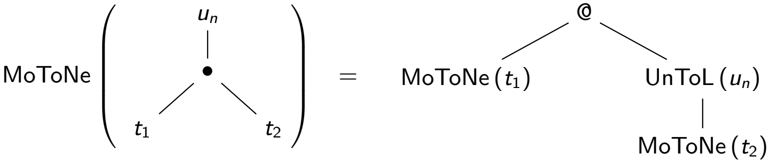
$$\mathsf{UnToL}\left(\begin{array}{c}\bullet\\ |\\ u_n\end{array}\right) = \begin{array}{c}\lambda\\ |\\ \mathsf{UnToL}(u_n)\end{array}$$

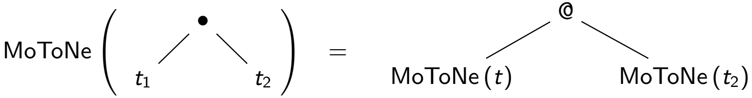
$$\mathsf{UnToD}(\bullet) = \mathtt{0}$$

$$\mathsf{UnToD}\left(\begin{array}{c}\bullet\\ |\\ u_n\end{array}\right) = \begin{array}{c}S\\ |\\ \mathsf{UnToD}(u_n)\end{array}$$

From Motzkin trees to neutral terms

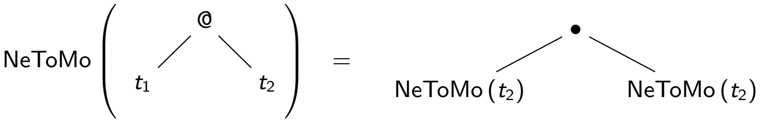
$$\mathrm{MoToNe}\,(u_n) \quad = \quad \mathrm{UnToD}\,(u_n)$$



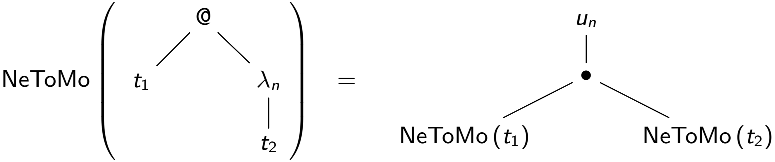


From neutral terms to Motzkin trees

$$\mathsf{NeToMo}\,(n) \quad = \quad u_n$$



where t_2 does not start with a λ



Summary

Theorem

There exist linear bijections among black-white trees, zigzag-free trees, and λ-terms.

Theorem

There exists a linear bijection between neutral λ-terms and Motzkin trees.

The end

Thank you.