On the Number of Lambda Terms With Prescribed Size of Their De Bruijn Representation

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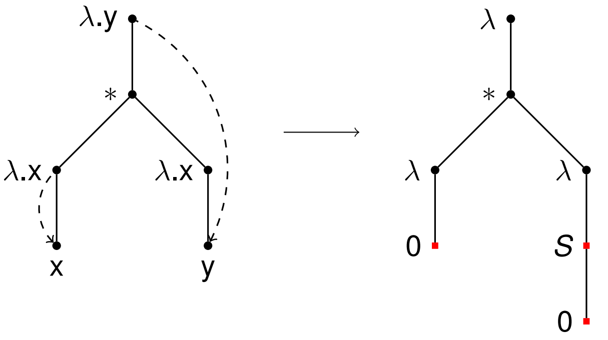
Definition of lambda terms

$$T ::= x \mid \lambda x.T \mid T * T \quad \rightarrow \quad T ::= S^n 0 \mid \lambda T \mid T * T$$

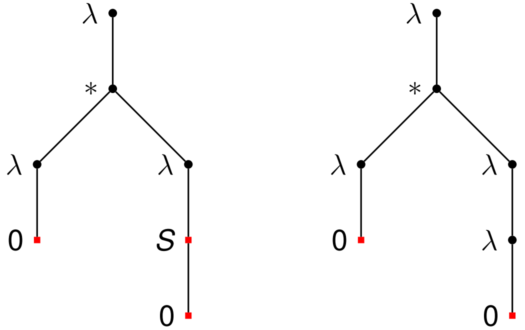
$\lambda x.T$: abstraction, unary node $\qquad (T * T)$: application, binary node

$$\lambda y.((\lambda x.x) * (\lambda x.y)) \rightarrow \lambda(\lambda 1 * \lambda 2) \rightarrow \lambda((\lambda 0) * (\lambda(S0)))$$



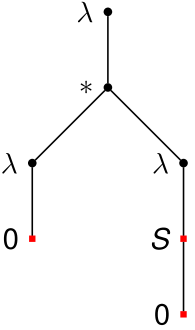
Definition of lambda terms

$$\lambda y.(\lambda x.x * \lambda x.y) \quad \neq \quad \lambda y.(\lambda x.x * \lambda x.\lambda z.z)$$
$$\lambda((\lambda 0) * (\lambda(S0))) \quad \neq \quad \lambda((\lambda 0) * (\lambda(\lambda 0)))$$

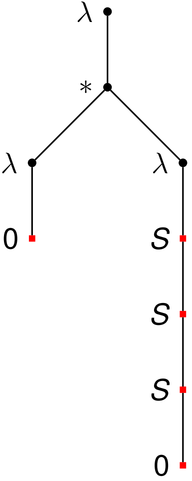


m-open lambda terms

closed lambda term (0-open)
$\lambda((\lambda 0) * (\lambda(S0)))$



2-open lambda term
$\lambda((\lambda 0) * (\lambda(SSS0)))$



General notion of size

$$\begin{aligned} |0| &= a \\ |S| &= b \\ |\lambda M| &= |M| + c \\ |MN| &= |M| + |N| + d. \end{aligned}$$

Assumptions

1. a, b, c, d *are nonnegative integers,*
2. $a + d \geq 1$,
3. $b, c \geq 1$,
4. $\gcd(b, c, a + d) = 1$.

General notion of size

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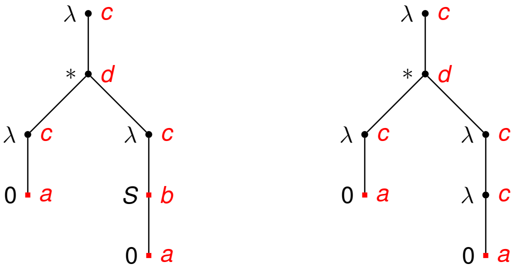

size: $2a + b + 3c + d$ size: $2a + 4c + d$

General notion of size

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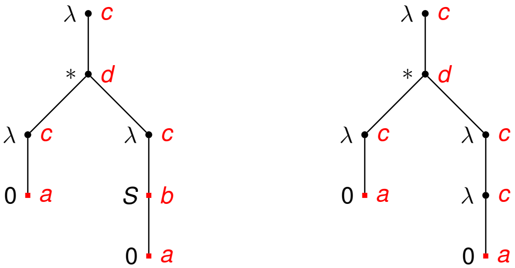

size: $2a + b + 3c + d$ size: $2a + 4c + d$

- natural counting (Bendkowski, Grygiel, Lescanne, Zaionc 2015): $a = b = c = d = 1$
- less natural counting (Bendkowski, Grygiel, Lescanne, Zaionc 2015): $a = 0$, $b = c = 1$, $d = 2$
- binary lambda calculus (Tromp 2006): $b = 1$, $a = c = d = 2$

Combinatorial specifications and generating functions

Let $(\mathcal{A}, |\cdot|_{\mathcal{A}})$, $(\mathcal{B}, |\cdot|_{\mathcal{B}})$ be comb. structures with generating functions

$$A(z) = \sum_{n \geq 0} a_n z^n = \sum_{x \in \mathcal{A}} z^{|x|_{\mathcal{A}}} \text{ and}$$

$$B(z) = \sum_{n \geq 0} b_n z^n = \sum_{x \in \mathcal{B}} z^{|x|_{\mathcal{B}}}.$$

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- If $\mathcal{A} \cap \mathcal{B} = \emptyset$ and $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ with

$$|x|_{\mathcal{C}} := \begin{cases} |x|_{\mathcal{A}} & \text{if } x \in \mathcal{A}, \\ |x|_{\mathcal{B}} & \text{if } x \in \mathcal{B}, \end{cases}$$

then $C(z) = A(z) + B(z)$.

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- If $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ and $|(a, b)|_\mathcal{C} = |a|_\mathcal{A} + |b|_\mathcal{B}$ then $C(z) = A(z) \cdot B(z)$.

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- If $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ and $|(a, b)|_{\mathcal{C}} = |a|_{\mathcal{A}} + |b|_{\mathcal{B}}$ then $C(z) = A(z) \cdot B(z)$.
- If $\mathcal{C} = \textsc{Seq}(A)$ and $|(a_1, \ldots, a_k)|_{\mathcal{C}} = \sum_{i=1}^{k} |a_i|_{\mathcal{A}}$ then $C(z) = \frac{1}{1-A(z)}$.

Combinatorial specification and lambda terms

$$\mathcal{L} = \text{SEQ}(\mathcal{S}) \times \mathcal{Z} \,\cup\, \mathcal{U} \times \mathcal{L} \,\cup\, \mathcal{A} \times \mathcal{L}^2$$

- $\mathcal{L}$ – the class of lambda terms,
- $\mathcal{Z}$ – the class of zeros,
- $\mathcal{S}$ – the class of successors,
- $\mathcal{U}$ – the class of abstractions,
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Remark: $\mathcal{Z}, \mathcal{S}, \mathcal{U}, \mathcal{A}$ contain only one atomic object.
Thus

$$L(z) = z^a \sum_{j=0}^{\infty} z^{bj} + z^c L(z) + z^d L(z)^2,$$

$[z^n]L(z)$=number of lambda terms of size n.

m-open terms and functional equations

Let

$$\mathcal{L}_m = \textsc{Seq}_{\leq m-1}(\mathcal{S}) \times \mathcal{Z} \;\cup\; \mathcal{U} \times \mathcal{L}_{m+1} \;\cup\; \mathcal{A} \times \mathcal{L}_m^2.$$

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- $L_{m,n}$ – the number of m-open lambda terms of size n,
- $L_m(z) = \sum_{n\geq 0} L_{m,n} z^n \quad ([z^n]\, L_m(z) = L_{m,n})$

$$L_m(z) = z^a \sum_{j=0}^{m-1} z^{bj} + z^c L_{m+1}(z) + z^d L_m(z)^2$$

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- $L_0(z)$ is the gen. fun. of the set $\mathcal{L}_0$ of closed lambda terms,
- $L_\infty(z)$ is the gen. fun. of the set $\mathcal{L}_\infty = \mathcal{L}$ of all lambda terms.

$L_\infty(z)$ – all terms

Solving

$$L_\infty(z) = z^a \sum_{j=0}^{\infty} z^{bj} + z^c L_\infty(z) + z^d L_\infty(z)^2.$$

we get

$$L_\infty(z) = \frac{1 - z^c - \sqrt{(1 - z^c)^2 - \frac{4z^{a+d}}{1-z^b}}}{2z^d},$$

which defines an analytic function in a neighbourhood of $z = 0$.

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Theorem (Flajolet, Odlyzko 1990)

If $\alpha \in \mathbb{R} \setminus \mathbb{N}$ and $f(z) \sim (1 - \frac{z}{\rho})^\alpha$ as $z \to \rho$ within a Δ-domain, then

$$[z^n]f(z) \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)} \rho^{-n}, \text{ as } n \to \infty.$$

$L_\infty(z)$ – all terms

Proposition

Let $\rho = \mathrm{RootOf}\left\{(1-z^b)(1-z^c)^2 - 4z^{a+d}\right\}$. *Then*

$$L_\infty(z) = a_\infty + b_\infty\sqrt{1-\frac{z}{\rho}} + \mathrm{O}\left(\left|1-\frac{z}{\rho}\right|\right),$$

for some constants $a_\infty > 0, b_\infty < 0$ *that depend on* a, b, c, d.

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Corollary

The coefficients of $L_\infty(z)$ *satisfy*

$$L_{\infty,n} \sim -\frac{b_\infty}{2\sqrt{\pi}}\rho^{-n}n^{-3/2}, \text{ as } n \to \infty.$$

Closed lambda terms

We are mainly interested in the asymptotics of the number $L_{0,n}$ of closed lambda terms.

Conjecture (Grygiel, Lescanne, 2015)

In the case of binary lambda calculus ($a = c = d = 2, b = 1$), for every m, $L_{m,n} = \mathrm{o}\left(n^{-\frac{3}{2}}\rho^{-n}\right)$.

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We will show that this conjecture is false.

Main Result

Theorem

Let $\rho = \mathrm{RootOf}\left\{(1-z^b)(1-z^c)^2 - 4z^{a+d}\right\}$. *Then there exist positive constants* $\underline{C}$ *and* $\overline{C}$ *(depending on* a, b, c, d *and* m*) such that the number of* m*-open lambda terms of size* n *satisfies*

$$\liminf_{n\to\infty} \frac{L_{m,n}}{\underline{C} n^{-\frac{3}{2}} \rho^{-n}} \geq 1 \quad \textit{and} \quad \limsup_{n\to\infty} \frac{L_{m,n}}{\overline{C} n^{-\frac{3}{2}} \rho^{-n}} \leq 1,$$

Remark

In case of given a, b, c, d *and* m *we can compute numerically such constants* $\underline{C}$ *and* $\overline{C}$*.*

Analyzing $L_m(z)$

We have

$$L_m(z) = z^a \sum_{j=0}^{m-1} z^{bj} + z^c L_{m+1}(z) + z^d L_m(z)^2.$$

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$\mathcal{L}_m^{(h)}$ – lambda terms in $\mathcal{L}_m$ where the length of each string of successors is bounded by h

$$L_m^{(h)}(z) = \begin{cases} z^a \sum_{j=0}^{m-1} z^{bj} + z^c L_{m+1}^{(h)}(z) + z^d L_m^{(h)}(z)^2 & \text{if } m < h, \\ z^a \sum_{j=0}^{h-1} z^{bj} + z^c L_h^{(h)}(z) + z^d L_h^{(h)}(z)^2 & \text{if } m \geq h, \end{cases}$$

because for $m \geq h$ we have $L_m^{(h)}(z) = L_h^{(h)}(z)$.

Lambda terms with bounded number of successors

For $m < h$

$$L_m^{(h)}(z) = \frac{1-\sqrt{r_m(z)+2z^c\sqrt{r_{m+1}(z)+2z^c\sqrt{\cdots\sqrt{r_{h-1}(z)+2z^c\sqrt{r_h(z)}}}}}}{2z^d},$$

where

$$r_j(z) = \begin{cases} 1-4z^{a+d}\frac{1-z^{jb}}{1-z^b} - 2z^c & \text{if } m \le j < h-1, \\ 1-4z^{a+d}\frac{1-z^{(h-1)b}}{1-z^b} - 2z^c + 2z^{2c} & \text{if } j = h-1, \\ (1-z^c)^2 - 4z^{a+d}\frac{1-z^{bh}}{1-z^b} & \text{if } j = h. \end{cases}$$

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Lemma

- *the dominant singularity* $\rho^{(h)} := \rho_m^{(h)}$ *of* $L_m^{(h)}(z)$ *is independent of m,*
- $L_m^{(h)}(z) \sim a_m^{(h)} + b_m^{(h)}\sqrt{1-\frac{z}{\rho^{(h)}}}$ *as* $z \to \rho^{(h)}$
- $\rho^{(h)} > \rho$ *as well as* $\lim_{h\to\infty}\rho^{(h)} = \rho$.

$\mathcal{K}_m$: the complement of $\mathcal{L}_m$

Let $\mathcal{K}_m = \mathcal{L}_\infty - \mathcal{L}_m$ and $K_m(z) = L_\infty(z) - L_m(z)$, then

$$K_m(z) = \frac{z^{a-cm}}{1-z^b} \sum_{j=m}^{\infty} z^{j(b+c)} \prod_{i=m}^{j} \frac{1}{1 - z^d \left(L_\infty(z) + L_i(z)\right)}.$$

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Lemma

For all $m \geq 0$ the radius of convergence of $K_m(z)$ equals ρ (the radius of convergence of $L_\infty(z)$).

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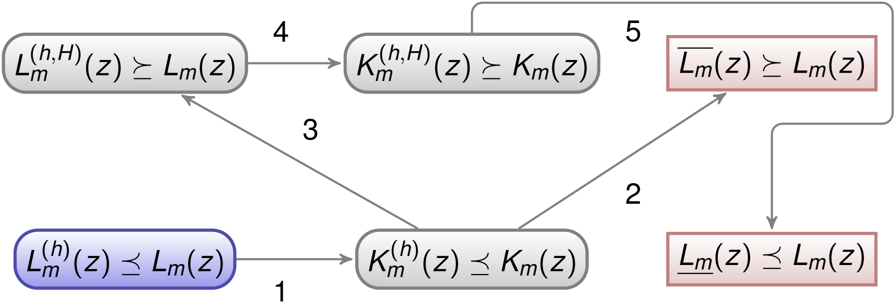
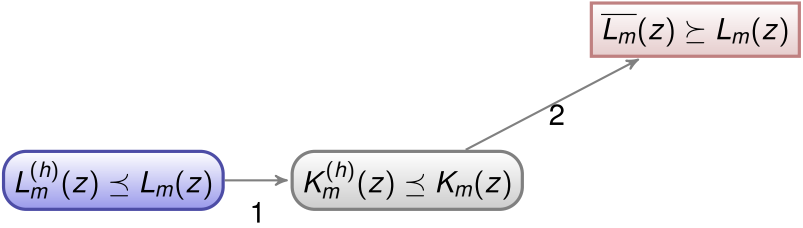
Lemma

All the functions $L_m(z)$, $m \geq 0$, have the same radius of convergence and it is equal ρ.



$L_m^{(h,H)}(z) \succeq L_m(z)$
4
$K_m^{(h,H)}(z) \succeq K_m(z)$
5
$\overline{L_m}(z) \succeq L_m(z)$
3
2
$L_m^{(h)}(z) \preceq L_m(z)$
$K_m^{(h)}(z) \preceq K_m(z)$
$\underline{L_m}(z) \preceq L_m(z)$
1

Upper bound for $L_{m,n}$



Upper bound for $L_{m,n}$

$$K_m^{(h)}(z) := z^a \sum_{j=m}^{\infty} z^{bj} + z^c K_{m+1}(z) + z^d K_m(z) L_\infty(z) + z^d K_m(z) L_m^{(h)}(z)$$

Lemma

The generating function $K_m^{(h)}(z)$ admits the expansion

$$K_m^{(h)}(z) = c_m^{(h)} + d_m^{(h)} \sqrt{1 - \frac{z}{\rho}} + \mathrm{O}\left(\left|1 - \frac{z}{\rho}\right|\right), \text{ as } z \to \rho.$$

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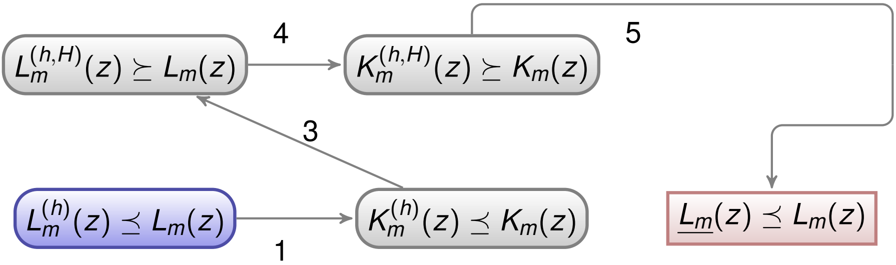
Consider $\overline{L_m}(z) = L_\infty(z) - K_m^{(h)}(z)$

Corollary

The number of m-open lambda terms of size n satisfies

$$\limsup_{n\to\infty} \frac{L_{m,n}}{\overline{C} n^{-\frac{3}{2}} \rho^{-n}} \leq 1 \ \textit{where } \overline{C} = \frac{b_\infty - d_m^{(h)}}{\Gamma\left(-\frac{1}{2}\right)}.$$

Lower bound for $L_{m,n}$



Lower bound for $L_{m,n}$

Define

$$L_m^{(h,H)}(z) = \sum_{n\geq 0} L_{m,n}^{(h,H)} z^n = \begin{cases} L_\infty(z) - K_m^{(h)}(z) & \text{if } m < H, \\ L_\infty(z) & \text{else.} \end{cases}$$

$$K_m^{(h,H)}(z) = z^a \sum_{j=m}^{\infty} z^{bj} + z^c K_{m+1}(z) + z^d K_m(z) L_\infty(z) + z^d K_m(z) L_m^{(h,H)}(z)$$

Lemma

The generating function $K_m^{(h,H)}(z)$ admits the following expansion

$$K_m^{(h,H)}(z) = c_m^{(h,H)} + d_m^{(h,H)} \sqrt{1 - \frac{z}{\rho}} + \mathrm{O}\left(\left| 1 - \frac{z}{\rho} \right| \right), \text{ as } z \to \rho.$$

Lower bound for $L_{m,n}$

Considering $\underline{L_m}(z) = L_\infty(z) - K_m^{(h,H)}(z)$ gives

Corollary

The number of m-open lambda terms of size n satisfies

$$\liminf_{n\to\infty} \frac{L_{m,n}}{\underline{C} n^{-\frac{3}{2}} \rho^{-n}} \geq 1 \textit{ where } \underline{C} = \frac{b_\infty - d_m^{(h,H)}}{\Gamma\left(-\frac{1}{2}\right)}.$$

Results for special notions of size

$$\begin{aligned} |0| &= a \\ |S| &= b \\ |\lambda M| &= |M| + c \\ |MN| &= |M| + |N| + d. \end{aligned}$$

- natural counting: $a = b = c = d = 1$,
- binary lambda calculus: $b = 1, a = c = d = 2$.

Natural counting

Lemma

The following bounds hold

$$\liminf_{n\to\infty} \frac{L_{0,n}^{(nat)}}{\underline{C}^{(nat)} n^{-\frac{3}{2}} \rho^{-n}} \geq 1 \quad \mathit{and} \quad \limsup_{n\to\infty} \frac{L_{0,n}^{(nat)}}{\overline{C}^{(nat)} n^{-\frac{3}{2}} \rho^{-n}} \leq 1$$

where $\rho = \mathrm{RootOf}\{-1 + 3x + x^2 + x^3\} \approx 0.295598\ldots$ *and* $\underline{C}^{(nat)}, \overline{C}^{(nat)}$ *are computable constants with numerical values* $\underline{C}^{(nat)} \approx 0.0779099{\color{red}5266\ldots}$ *and* $\overline{C}^{(nat)} \approx 0.0779099{\color{red}8229\ldots}$.

Binary lambda calculus

Lemma

The following bounds hold

$$\liminf_{n\to\infty} \frac{L_{0,n}^{(bin)}}{\underline{C}^{(bin)} n^{-\frac{3}{2}} \rho^{-n}} \geq 1 \quad and \quad \limsup_{n\to\infty} \frac{L_{0,n}^{(bin)}}{\overline{C}^{(bin)} n^{-\frac{3}{2}} \rho^{-n}} \leq 1$$

where $\rho = \mathrm{RootOf}\{-1 + x + 2x^2 - 2x^3 + 3x^4 + x^5\} \approx 0.509308\ldots$ *and* $\underline{C}^{(bin)}, \overline{C}^{(bin)}$ *are computable constants with numerical values* $\underline{C}^{(bin)} \approx 0.01252417\ldots$ *and* $\overline{C}^{(bin)} \approx 0.01254593\ldots$.

Thank you!