Quantitative aspects of normal-order reduction in combinatory logic

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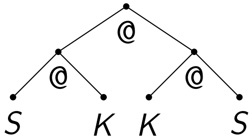
Size notion



$$SK(KS)$$

$$|t| = 3$$

Counting of SK-terms

$$C = C\,C \;\oplus\; S \;\oplus\; K$$

$$\downarrow$$

$$C(z) = zC^2(z) + 2$$

$$C(z) = \frac{1 - \sqrt{1-8z}}{2z}$$

Counting SK-terms

$$[z^n]C(z) = \frac{2^{n+1}}{n+1}\binom{2n}{n}$$

$$[z^n]C(z) \sim 8^n \frac{2}{\sqrt{\pi} n^{3/2}}$$

Normal-order reduction

$$Sxyz \to xz(yz)$$

$$Kxy \to x$$

Theorem (Curry, Feys 1958)

If x has a normal-form, then reducing x in *normal-order*, i.e. leftmost outermost redex first, will eventually lead to its normal form.

Previous results

Theorem (Bendkowski, Grygiel, Zaionc 2015)

The set of weakly normalizing SK-terms $\mathcal{WN}$ cannot have a trivial $0 - 1$ asymptotic density in the set of all combinators. In particular, if it exists, then

$$^{1}/_{32} \leq \mu\left(\frac{\mathcal{WN}}{C}\right) \leq 1 - {}^{1}/_{2^{18}}$$

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$$0.03125 \leq \mu(\frac{\mathcal{WN}}{C}) \leq 0.999996$$

Feasible constructions

$$^1/_{32} \longrightarrow KKC \oplus KSC$$

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$$1/32 \longrightarrow KKC \oplus KSC$$

$$1 - 1/2^{18} \longrightarrow \Omega_1 \, \mathrm{SEQ}(C) \oplus \Omega_2 \, \mathrm{SEQ}(C)$$
$$\Omega_1 = S(SS)SSSS \quad \Omega_2 = SSS(SS)SS$$

Our contributions

Theorem (Bendkowski 2016)

There exist computable unambiguous regular tree grammars $(R_n)_{n\in\mathbb{N}}$ defining SK-terms reducing in exactly n normal-order reduction steps.

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Theorem (Bendkowski 2016)

$$\mathbf{0.34010} \leq \mu^{-}(\frac{\mathcal{WN}}{C})$$

...and pending!

Normal-order reduction grammars

Key idea

Suppose that x reduces in n normal-order reduction steps to its normal form. Let y be the term such that $x \to y$. Then y reduces in just $n - 1$ steps. Hence, R_{n+1} should depend structurally on R_n.

Normal-order reduction grammars

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Start with normal forms:

$$R_0 = S \oplus K \oplus KR_0 \oplus SR_0 \oplus SR_0R_0$$

K-Expansions: The easy part...

Let $y \in L(X\alpha_1 \ldots \alpha_m)$ where $X\alpha_1 \ldots \alpha_m \in R_n$. Suppose that $x \to y$ and moreover $x = Kx_1 \ldots x_k$. What production in R_{n+1} defines x?

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$$K(X\alpha_1 \ldots \alpha_m)C$$

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$$\ldots$$

$$KXC\alpha_1 \ldots \alpha_m.$$

S-Expansions: The hard part...

Let $y \in L(X\alpha_1 \ldots \alpha_m)$ where $X\alpha_1 \ldots \alpha_m \in R_n$. Suppose that $x \to y$ and moreover $x = Sx_1 \ldots x_k$. What production in R_{n+1} defines x?

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$$x = S(Xy_1 \dots y_d)x_2x_3 \dots x_k \to Xy_1 \dots y_d x_3(x_2x_3)x_4 \dots x_k = y$$

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Problem: find out whether α_{i+1} 'rewrites' to some $\beta = \gamma\alpha_i$.

Production rewriting

If α has a production β, or α is the set C of all SK-combinators, then α *rewrites* to β. Such an operation limits α's language, since if α rewrites to β, then $L(\beta) \subseteq L(\alpha)$.

Mesh Set

Input: Non-rewritable α and β.
Output: A complete partition of $L(\alpha) \cap L(\beta)$ using γ such that both α and β rewrite to γ.

Rewriting Set

Input: α and β.

Output: All possible γ and η such that α rewrites to γ and β rewrites to $\eta\gamma$.

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Example

$$\textsc{RewritingSet}(S, R_0) = \{SS, KS, SR_0S\}$$

Normal-order reduction grammars

$$
\begin{aligned}
R_0 &= S \oplus K \oplus KR_0 \oplus SR_0 \oplus SR_0 R_0 \\
R_1 &= SR_1 \oplus KR_1 \oplus SR_0 R_1 \oplus SR_1 R_0 \oplus KR_0 C \oplus \\
&\quad KSCR_0 \oplus KKCR_0 \oplus KSCR_0 R_0 \oplus K(SR_0)CR_0 \oplus \\
&\quad SSSR_0 \oplus SSKR_0 \oplus SS(SR_0)R_0 \\
R_2 &= \ldots
\end{aligned}
$$

Normal-order reduction grammars

n	$\lvert R_n \rvert$
0	5
1	12
2	75
3	625
4	5673
5	53164
6	508199
7	4926651

Upper bound

Theorem (Bendkowski 2016)

There exists a primitive recursive function $\psi\colon \mathbb{N} \to \mathbb{N}$, upper bounding the number of R_n's productions, i.e. $|R_n| \leq \psi(n)$.

Upper bound

n	$\lvert R_n \rvert$	$\psi(n)$
0	5	5
1	12	$\approx 6 \cdot 10^{84549}$
2	75	?
3	625	?
4	5673	?
5	53164	?
6	508199	?
7	4926651	?

Asymptotic aspects of R_n

Theorem (Bendkowski 2016)

Each grammar R_n has a corresponding computable ogf. Moreover, for each $n \geq 1$ the ogf $R_n(z)$ has a removable singularity at $z = 0$ and a single dominating singularity on its circle of convergence $\rho_n = 1/8$.

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Corollary

For each $m \geq 1$, we have

$$[z^n]R_m(z) \sim 8^n \frac{\bar{C}_m}{-2\sqrt{\pi} n^{3/2}}.$$

Asymptotic density of R_n in C

n	$\mu\left({}^{R_n}/_{C}\right)$
0	0.
1	0.08961233291075565
2	0.06417374404832035
3	0.0501056553007704
4	0.04131967414765603
5	0.03570996929825453
6	0.03119525702124082
7	0.027987393260263862

Symbolic algebra is tough. . .

Why stop at $n = 7$? Can't we compute R_8, R_9, R_{10}, . . .?
We **could**, however computing R_7 takes $\sim$ 20 GB RAM.
Mathematica computations of $\mu({R_7}/{c})$ take $\sim$ 52 GB RAM!

Super-computer experiments

- Sample s uniformly random SK-terms of size n.
- Evaluate each sample using up to r normal-order reduction steps. If the sample reduced to its normal form, write down the actual number of reduction steps it took. Otherwise, write down -1.

$s = 1200$ samples, $\leq r = 1000$ reductions, size $n = 50000000$

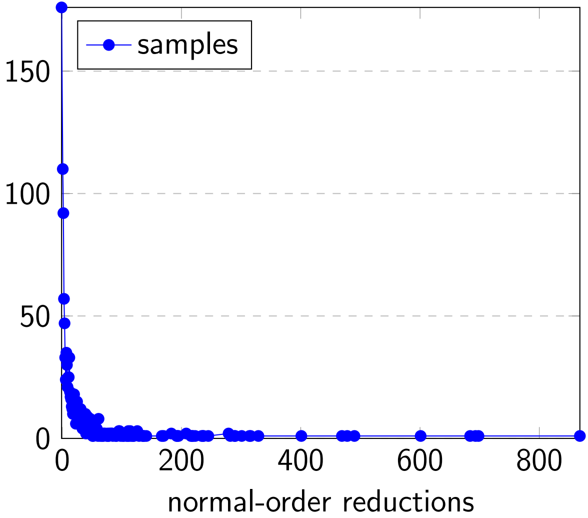

Figure: $E(X) \approx 31.5810$, $\log_2 n \approx 25.5754$

Open problems

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Experiments suggest that

$$\mu(\frac{\mathcal{WN}}{C}) \approx 0.84.$$

Is it true?

Open problems

We know that

$$\sum_{n=0}^{\infty} \mu(\frac{R_n}{C}) \quad \text{exists.}$$

Hence, the following problem.

Problem

$$\sum_{n=0}^{\infty} \mu(\frac{R_n}{C}) \stackrel{?}{=} \mu(\frac{\bigcup_{n=0}^{\infty} R_n}{C})$$

Questions and (possibly) answers

Thank you for your attention!