Cayley Monoids

Introduction

- In 1937 Church formulated lambda calculus as a semigroup;
- more precisely, with eta conversion, as a monoid. Conversely,
- it is natural to ask when a monoid can support an application
- operation. Indeed, in 1975 Dana Scott said that we should view
- combinatory logic as "combinatory algebra";so, let us try this.
- In this note we propose a possible modus operandi; namely, the
- notion of a Cayley monoid.
- Everyone is familiar with Cayley's regular representation
- of groups in the symmetric group. It is clear that it applies in
- a limited way to monoids. The notion of a Cayley monoid is just
- an internalization of this type of representation.

Cayley meets Church

Definition: A Cayley monoid K is a structure (M,*,i,a,b,A,B) where

(1) (M,*,i) is a monoid

(2) a,b : M

(3) A : M -> M and B : M -> M such that for all x,y,z:M

(i) A(a * B(x)) = A(x)

(ii) A(b * B(x)) = B(x)

(iii) A(i * B(x)) = x

(iv) A(x * y * B(z)) = A(x * B(A(y * B(z)))

Notation: Let K be a Cayley monoid. For each x:M we can define
X : M -> M by X(u) = A(x * B(u)).

Example 1: For any monoid M there is always the trivial Cayley monoid A(x) = B(x) = x, a = b = i. Here X(u) = x*u.

Example 2: If g belongs to the group of a monoid M with inverse
g^ then we can set b := g, a := g^ A(x) = a^ * x and B(x) = a * x.
Just as in example 1 we get a Cayley monoid K . A concrete example is given the "almost" isometries .If u,v are vectors in the plane R x R let E(u,v) be the Euclidian distance of v from u.
f : R x R -> R x R
is said to be an almost isometry if there exists e in R such that
(1) | E(f(u),f(v)) - E(u,v) | < e, and
(2) For each v there exists u s.t. E(f(u),v) < e.
The almost isometries form a monoid under composition of maps. Now
set b(u) = 2u, a(u)= 1/2 u, B(f) = 2f and A(f) = 1/2 f so we have a Cayley monoid K.

Let O be a subset of M
Definition: The Cayley monoid K is said to have an autonomous commutator relative to O if there exists c : M
such that whenever x and y are distinct members of O
we have
(v)_{O} A(A(c * B(x)) * B(y)) = A(y * B(x)).

Example 1 continued: Let M be the group of quaternions
{p,q,r,-p,-q,-r,i,-i}
We have used the letters 'p','q','r' for the usual 'i','j','k’.
Let O = {p,q,r}.We have the autonomous commutator property for c = -i

Example 3: Let M be the monoid of all functions from the set of all non negative reals into itself. As in example 2, let
$g(x) := k + x$ for k a positive integer
and let
$f_{n}(x) := n - e^{-x}$.
We set $O = \{ f_{n} \mid n \text{ a positive integer} \}$. Now c is defined as follows:
Input x
solve $x = n - e^{-y} - k$
solve $y = (m - e^{-z}) + k$
Output
$n - e^{ m - e^{-z} + k } = k$

Example 4: For any pre-complete Ershov numbering of N, where ~ is complete for 0’, the monoid of morphisms supports a Cayley monoid, where there is an autonomous commutator (general case?)

Definition : In a Cayley monoid K we say that B is an endo if

For all x,y:K we have B(x*y) = B(x)*B(y).

and B(i) = i.

Example 5:

The Freyd-Heller monoid has been rediscovered many times
It is the positive part of Thompson's group F,
and can be presented with the infinite set of generators b_{n}
for natural numbers n and the relations

b_{n+1} * b_{k} = b__{k} * b__{n}

for k < n. This is also a presentation of a monoid. Here we
wish to add left inverses a_{n} satisfying a_{n}*b_{n} = i
and

a_{k} * b_{n+1} = b_{n} * a_{k}
a_{k} * a_{n+1} = a_{n} * a_{k}

for k < n+1. This is not yet the group F but another monoid M.

Now define

$$B(b_{n}) = b_{n+1}$$

$$B(a_{n}) = a_{n+1}$$

then B extends to an endomorphism

$$B(x*y) = B(x)*B(y).$$

So, given these relations each element can be written in the unique normal form

$$B(d) * b^{k} * a^{l}.$$

This requires proof ; more about this later.

Now M has a Cayley monoid structure where B is as above and A is defined by

$$A(B(d) * b^{k} * a^{l}) = d.$$

Fact : In this Cayley monoid

$$x * y = A(A(b * B(x)) * B(y))$$

Proof: $A(A(b * B(x)) * B(y)) = A(B(x) * B(y))$

$= A(B (x * y))$

$= x * y.$

The relations of M are
realized by the linear lambda calculus under beta-eta conversion,
here denoted '~', with
b = \xyz. x(yz) (the combinator 'B')
a = \x. xI (the combinator 'CII')
x*y = bxy (the combinatory semigroup).
Below we shall refer to this Cayley monoid as
R(ichard)A(lex)P(eter)
In RAP we have

(1)B(B(d)) * b = b * B(d) for all d:M

(2)a * b = i

Our representation of RAP in linear lambda calculus gives

(3) C(x) * B(y) = y * C(x)

(4) C(x) * c = x (Church)

(5) C(A(x * B(y)) = C(y) * C(x) * b (Church)

(6) A(c) = a

Theorem: If K is a Cayley monoid with the endo property,
an autonomous commutator, and (2)-(6) then K
contains a copy of the linear
lambda calculus with B = b and J = c

Proof: the proof is like an axiomatized version of chapter 7 of Barendregt's book. It gives us a homomorphism of the linear lambda calculus into K. For faithfulness, we need a version of Bohm's theorem for linear lambda calculus (known?)

Theorem: Every monoid can be embedded into a Cayley
Monoid with the endo property, an autonomous
commutator, and (2)-(6).

Proof: We embed M into the lambda calculus using the Hindley-Rosen Theorem.

Problem:

Can we define a Cayley monoid with an autonomous commutator on the set of all functions R -> R?

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