# Almost all classical propositions are intitionistic

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# Zaionc paradox

In 2007, in two papers Zaionc and co-authors addressed the following paradox.

*Asymptotically almost all classical propositions are intionistic.*

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In this paper, I consider experimentally what almost all means.

This cannot be all, since $((\alpha \to \beta) \to \alpha) \to \alpha$ is classical, but not intutionistic.

# Zaionc paradox

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Genitrini, Kozik, and Zaionc consider a data structure which I call canonical expressions.

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Genitrini, Kozik, and Zaionc consider a data structure which I call canonical expressions.

In this work : I address those *canonical expressions*, namely

- implicative propositions are considered up-to renaming of variables,
- two implicative propositions are the same if they differ by the renaming of variables.
- better, I consider implicative propositions with a canonical naming of variables.

# Restricted Growth Strings

Consider the list of variables of an expression of size 10 :

$$x \quad y \quad y \quad x \quad y \quad x \quad z \quad x \quad x \quad x$$

or

$$\beta \quad \alpha \quad \alpha \quad \beta \quad \alpha \quad \beta \quad \gamma \quad \beta \quad \beta \quad \beta$$

If we name canonically the variables, we get :

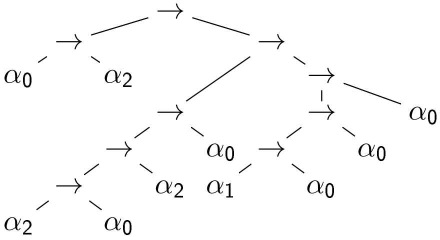
$$\alpha_0 \quad \alpha_2 \quad \alpha_2 \quad \alpha_0 \quad \alpha_2 \quad \alpha_0 \quad \alpha_1 \quad \alpha_0 \quad \alpha_0 \quad \alpha_0$$

which corresponds to the string of natural

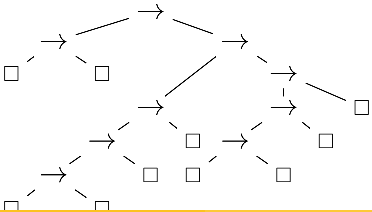
$$0220201000$$

- Restricted Growth Strings for Knuth,
- Irregular Staircases for Flajolet and Sedgewick and
- more generally equivalence classes.

Canonical expressions are binary expressions with variables named canonically by a restricted growth string. The binary operator is $\rightarrow$.



Hence a canonical expression is a pair (binary tree, restricted growth string)



$, \qquad 0220201000$

# Catalan and Bell numbers

- Binary trees are counted by Catalan numbers, $C_n$.
- Restricted growth string or equivalence classes are counted by Bell numbers, $\varpi_n$.
- Canonical expressions are counted by sequence $K_n = C_{n-1}\varpi_n$ (A289679 in the OEIS).

$$K_n \quad \sim \quad n! \frac{4^{n-1} e^{e^r - 1}}{\pi \sqrt{2(n-1)^3 r(r+1) e^r}}$$

$r \equiv r(n)$ is the positive root of the equation $re^r = n + 1$.

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- I generate random objects of a given size,
- I check properties on those objects,
- I count objects fulfilling those properties,
- I collect statistics for objects of that size.

This relies on

- an efficient and correct random generation algorithm and
- efficient algorithms for checking properties, at the price of checking only weak properties.

# Random generation

There are two linear algorithms for random generation found in Knuth's books
The Art of Computer Programming chapter 4.

- Remy's algorithm for random generation of binary trees,
- Stam's algorithm for random generation of equivalence classes.

# Ratio of simple intuitionistic theorems

$R_n$ is the ratio of simple intuitionistic theorems (goal as premise) among canonical expressions.

| $n$ | $\frac{log(n)}{n}$ | $R_n$ |
|---|---|---|
| 25 | $0,128755033$ | 0.2214 |
| 50 | $0,07824046$ | 0.1248 |
| 100 | $0,046051702$ | 0.0506 |
| 500 | $0,012429216$ | 0.0119 |
| 1000 | $0,006907755$ | 0.006 |

# Ratio simple theorems vs non simple antilogies

- Simple theorems are propositions with the goal as a premise, as a first approximation of intuitionistic theorems.
- Simple antilogies are propositions of the form

$$\ldots \to (\ldots \to \ldots \to x_i) \to \ldots \to x_0 \quad \textit{with} \quad x_i \neq x_0$$

as an approximation of non classical tautologies aka antilogies.

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This is the way Genitrini, Kozic and Zaionc approximate intuitionistic / classical ratio.

*36% non simple antilogies are simple theorems.*

More precisely,

*Among 10 000 random canonical expressions I found 238 simple theorems for 685 non simple antilogies.*

# MP theorem

An MP theorem is a proposition

- with goal $\alpha_i$ and
- two premises $\alpha_j$ and $\alpha_i \to \alpha_j$

Therefore it has the form :

$$... \to (\alpha_j \to \alpha_i) \to ... \to \alpha_j \to ... \to \alpha_i$$

or

$$... \to \alpha_j \to ... \to (\alpha_j \to \alpha_i) \to ... \to \alpha_i$$

An easy theorem is a theorem which is simple or MP.

# Minor, Cheap

**Removing easy premises**

In intuitionistic logic if a premise is a theorem, it can be removed.

I remove easy premises.

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### Minor theorems

A minor theorem is a theorem of the form $\ldots \to p \to \ldots \to p$.
$p$ is a proposition.

Minor premises are detected only when easy subexpressions have been removed.

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$p$ is a proposition.

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**Cheap theorems**

Cheap theorems are expressions that are minor or easy after removing (recursively) easy premises.

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On canonical expressions of size 100,

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- *On a sample of 20 000 random canonical expressions,*
- *I found 759 classical tautologies,*
- *among which 733 are cheap theorems.*

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On canonical expressions of size 100,

*96% classical tautologies are cheap theorems (intuitionistic theorems).*

More precisely

- *On a sample of 20 000 random canonical expressions,*
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The status of the 26 (759 - 733) propositions in between is not certain.

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- Analytic Combinatorics, should confirm why 96% of classical propositions are cheap theorems for size 100 and even more! ☺

Thank you for your attention