Holonomic equations and efficient random generation of binary trees

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Holonomic equation

An holonomic recurrence is

$$P_s(n)F_{n+s} + P_{s-1}(n)F_{n+s-1} + \ldots + P_0(n)F_n = 0$$

where the $P_s(n)$ are polynomials in n.

The paradigm is

$$(n+1)C_n - 2(2n-1)C_{n-1} = 0$$

an equation for Catalan numbers known from *Olinde Rodrigues* in 1838.

Catalan, Motzkin, Schröder

Numbers

Catalan

$$(n+1)C_n = 2(2n-1)C_{n-1}$$

counts binary trees.

Motzkin

$$(n+2)M_n = (2n+1)M_{n-1} + 3(n-1)M_{n-2}$$

counts unary binary trees.

Schröder

$$3(2n-1)S_n = (n+1)S_{n+1} + (n-2)S_{n-1}$$

counts binary trees in which every nonnull right link is colored either white or black.

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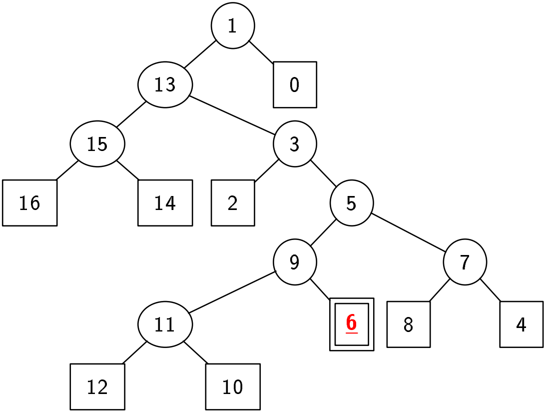
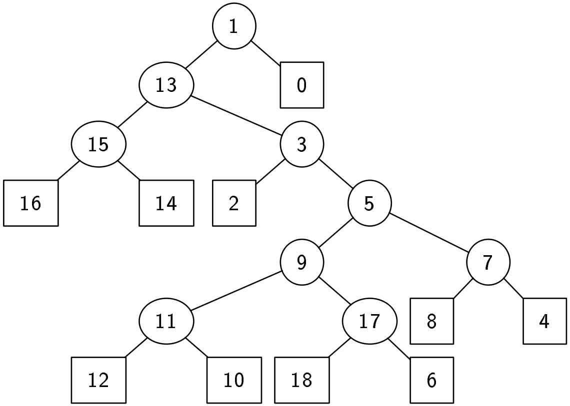
Rémy's construction



Implementation in an array of size $2n + 1$



- Nodes are labeled by odd numbers
- Leaves are labeled by even numbers

indices	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
values	1	13	0	2	5	9	7	8	4	11	17	12	10	15	3	16	14	18	6

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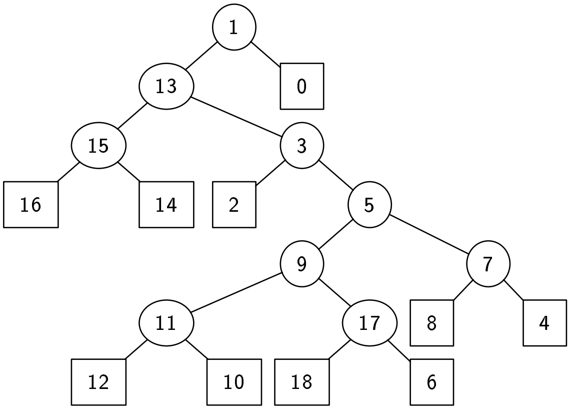

- Nodes are labeled by odd numbers
- Leaves are labeled by even numbers
- root is index 0
- left child of node labeled by $2n+1$ is at index $2n+1$
- right child of node labeled by $2n+1$ is at index $2n+2$.

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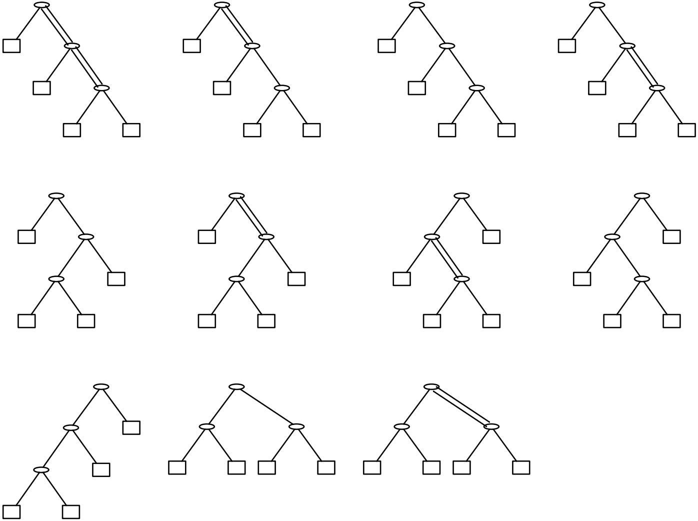
Schröder

$$(n+1)S_{n+1} = 3(2n-1)S_n - (n-2)S_{n-1}$$

In the construction,

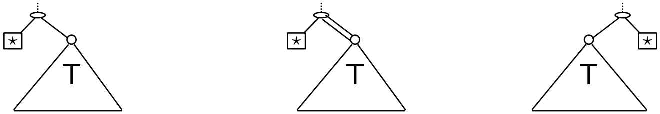
- one builds a Schröder tree of size $n+1$ from a tree of size n, or
- one fails.

Foata & Zeilberger construction for Schröder trees



The 11 Schröder trees with 4 leaves.

Insertion of a leaf in a Schröder tree



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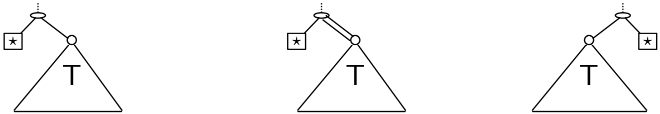
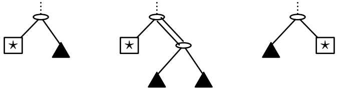


Three impossible insertions of leaves



a *b* *c*

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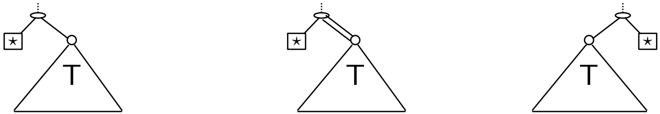

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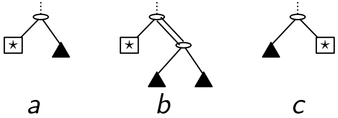
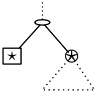


Two unreachables

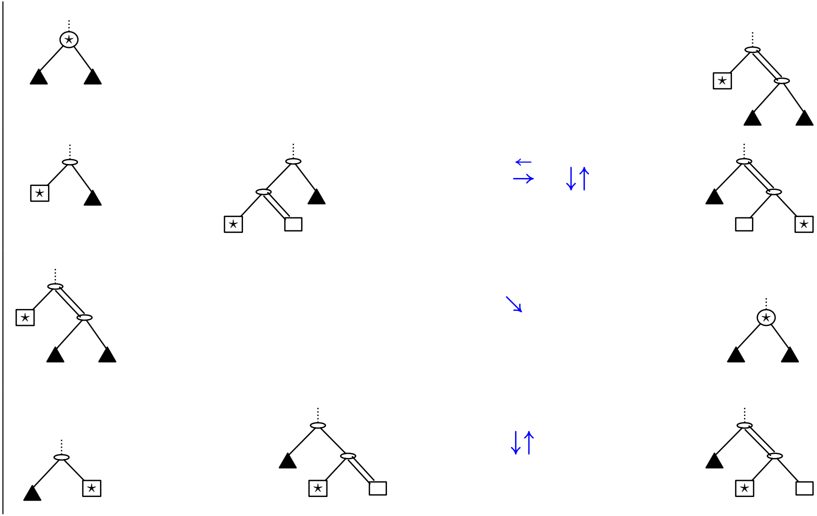
Foata-Zeilberger Isomophism (first case)

$\mathbf{L_1}$ 



Foata-Zeilberger Isomophism (second case)

$\mathbf{L_2}$



Foata-Zeilberger Isomophism (third case)

$\mathbf{R_1}$

The data structure

Like for Rémy's algorithm, one uses an array of size $2n+1$.

To represent colors of the links, one adds a boolean component.

- A Node is labeled by a pair of an odd number and a boolean.
- A Leaf is labeled by a pair of an even number and a boolean.

The boolean says that the right link that starts from this node is white.

Therefore when one considers a triple $(m,(k,b))$:

- m is an index (for a right link),
- (k,b) is located at m in the array (k corresponds to a node).

If $b \equiv \mathit{True}$ then m is even and k is odd.

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This must be checked when designing the algorithm.

The algorithm (1)

6 cases L_1, L_2 (with 4 subcases), R_1.

Draw a number x between 0 and $6n - 4$.

- L_1 if $x \textbf{ mod } 3 \equiv 0$
- L_2 if $x \textbf{ mod } 3 \equiv 1$
- R_1 if $x \textbf{ mod } 3 \equiv 2$.
- Let us call k the number $x \div 3$.

Assume the k^{th} is (h, b).

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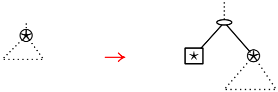

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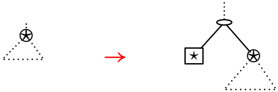

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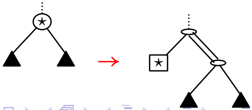

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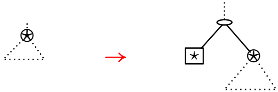

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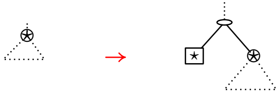

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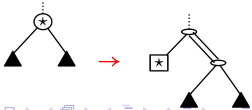

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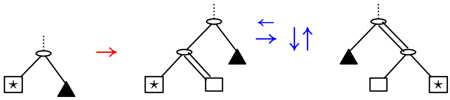

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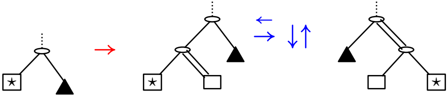

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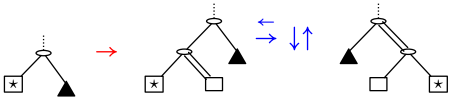

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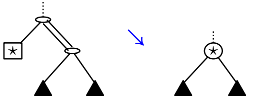
Failure **Retry**



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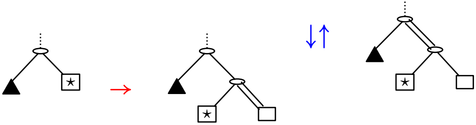

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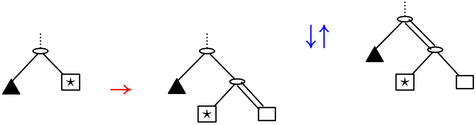

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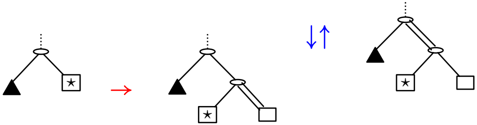

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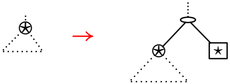
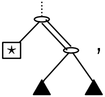


Complexity

The algorithm is quasi-linear.

- Except for  , all the cases require a computation in $O(1)$ to build a tree of size n from a tree of size $n-1$.

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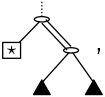
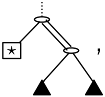
- Except for  , all the cases require a computation in $O(1)$ to build a tree of size n from a tree of size $n-1$.
- In case  , one fails and retries with probability less that $\frac{1}{3}$ and the total average complexity of building a tree of size n is in $O(n)$.

Benchmarks

size	time	ratio
1 000	$0.012s$	0.024
5 000	$0.031s$	0.0288
10 000	$0.064s$	0.025
50 000	$0.200s$	0.0269
100 000	$0.290s$	0.02707
500 000	$1.295s$	0.027762
1 000 000	$3.065s$	0.027883
5 000 000	$15.183s$	0.0276378
10 000 000	$30.738s$	0.0275827

Thank you !

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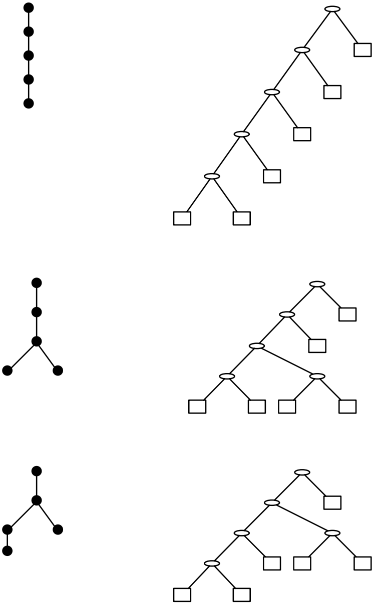
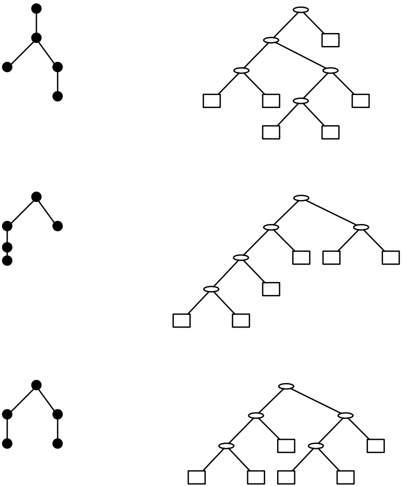
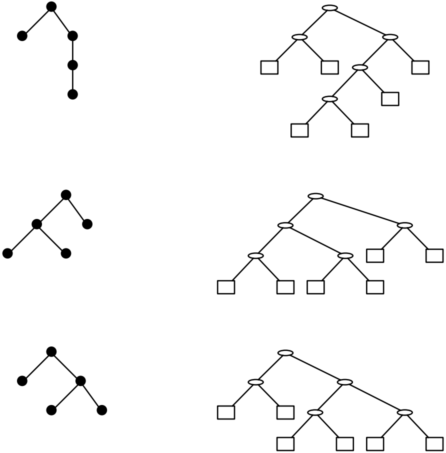
Any Question ?

The 9 Motzkin trees and the slanted binary trees

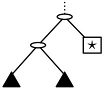
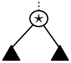
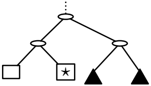
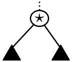
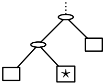
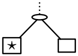
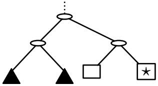
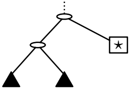
Instead of Motzkin trees, we consider slanted binary trees.

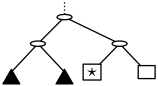
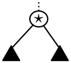
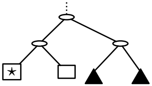
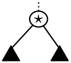
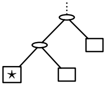
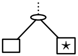






The 7 patterns of leaf-marked slanting trees

	leaf-marked slanting trees	marked slanting trees	Choice and node-marked slanting trees
1.			
2.			LR , 
3.			
4.			RR , 

5.			**RL** , 
6.			**LL** , 
7.			

The key choice

In the algorithm there are two cases

1. Building a tree of size n from a tree of size $n-1$,
2. Building a tree of size n from a tree of size $n-2$,

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1. Building a tree of size n from a tree of size $n-1$,
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Assume we draw a number between 0 and 1, and

- if $c \leq \frac{(2n+1)Mn-1}{(n+2)M_n}$, we choose case1,
- if $c > \frac{(2n+1)Mn-1}{(n+2)M_n}$, we choose case2.

For an implementation with no recursion

For an imlementation with a while loop, I proceed as follows :

1. I create the stack of recursive calls,

For an implementation with no recursion

For an imlementation with a while loop, I proceed as follows :

1. I create the stack of recursive calls,
2. I pop the stack, building the Motzkin trees from the small ones to the large ones.

I can build a random Motzkin tree of size 10 millions in 45s.