Holonomic equations and efficient random generation of binary trees

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Considering a recurrence defining a sequence of integer coefficients F_n. In this paper, I am interested in specific recurrences called "holonomic recurrence" where roughly speaking, "holonomic" means that F_{n+s} is a combination, using polynomials in n, of the F_i's, for $n \leq i \leq n+s$. More precisely, (see Flajolet and Sedgewick's book, Appendix B.4) the coefficients fulfill the following recurrence:

$$P_s(n)F_{n+s} + P_{s-1}(n)F_{n+s-1} + ... + P_0(n)F_n = 0$$

for some $n \geq n_0$, where the $P_j(n)$ are polynomials in n. This kind of recurrence is called a *P-recurrence*. For instance, for Catalan numbers:

$$C_n - \sum_{k=0}^{n-1} C_k C_{n-k-1} = 0$$

is the classical recurrence that is used in general to defined them, but it is not a P-recurrence, whereas

$$(n+1)C_n - 2(2n-1)C_{n-1} = 0$$

is the P-recurrence, which will be considered later on. Notice that initial values should be added to this P-recurrence. This will be considered in the paper for each specific case.

In this paper, I consider three families of binary trees (binary trees, Motzkin trees aka unary-binary trees, Schröder trees) and their random generation. It turns out that holonomic recurrences play a key role in the design of efficient random generation algorithms.

The three examples: binary trees, Motzkin trees, Schröder trees are interesting because they have different holonomic equations, one (Catalan numbers) has one term on the right, one (Motzkin numbers) has a sum of two terms on the right and one (Schröder numbers) has a subtraction of two terms, on the right. These yield different algorithms, as this is explained further in this paper.

This paper is associated with a Haskell program available on GitHub which serves as an executable specification. The full paper can be loaded here.