The categorical logic of elementary arithmetic

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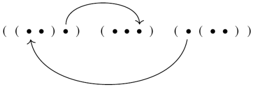


Au fond de l'Inconnu pour trouver du nouveau!

A little light reading

- On strict extensional reflexivity in compact closed categories
  Outstanding Contributions in Logic, (to appear March 2023)
  https://arxiv.org/abs/2202.08130
- The inverse semigroup theory of elementary arithmetic
  (submitted 2022)
  www.arxiv.org/abs/2206.07412
- From a conjecture of Collatz to Thompson's group F, via a conjunction of Girard
  (submitted 2022)
  www.arxiv.org/abs/2202.04443
- Congruential functions via categorical coherence, *(submitted 2022)*
- Operads, Groups, Monoids, & Conjectures. *(Draft manuscript)*

FRACTRAN – syntax and semantics

A programming language introduced by John H. Conway (1987)

Syntax Programs are (finite?) lists of positive rationals :

$$\left[\frac{P_0}{Q_0} , \frac{P_1}{Q_1} , \frac{P_2}{Q_2} , \frac{P_3}{Q_3} , \ldots \right]$$

Execution **Input** is a positive natural number $n \in \mathbb{N}^+$

The iterated step :

- Multiply n by each $\frac{P_0}{Q_0}, \frac{P_1}{Q_1}, \frac{P_2}{Q_2}, \frac{P_3}{Q_3}, \ldots$ in turn, until a whole number $n \times \frac{P_j}{Q_j} \in \mathbb{N}^+$ is found.
- replace n by this number $n \leftarrow n \times \frac{P_j}{Q_j}$

Conditional looping The above step is repeated, until the end of the list is reached.

Output Either :

1. The final value of n.
2. The sequence of values n takes on during execution.

An example FRACTRAN program

FRACTRAN is computationally universal

— it can simulate e.g. Turing Machines, or pure untyped λ-calculus

Conway gave the following example :

$$\left[\frac{17}{91}, \frac{78}{85}, \frac{19}{51}, \frac{23}{38}, \frac{29}{33}, \frac{77}{29}, \frac{95}{23}, \frac{77}{19}, \frac{1}{17}, \frac{11}{13}, \frac{13}{11}, \frac{15}{2}, \frac{1}{7}, \frac{55}{1} \right]$$

On input $n = 2$, this program fails to terminate[1]. The powers of two in the infinite sequence generated are :

$$2^2, \ldots, 2^3, \ldots, 2^5, \ldots, 2^7, \ldots, 2^{11}, \ldots, 2^{13}, \ldots, 2^{17}, \ldots, 2^{19}, \ldots, 2^{23}, \ldots, 2^{29}, \ldots$$

We recover the list : $[2^p \ : \ p \in \textbf{Primes}]$.

Fun exercise : What is the action of the program given by i/ *Inverting every fraction*, ii/ *reversing the list*, iii/ *doing both* ??

[1] First proved by Euclid, 350 BCE

Registers and conditionals from p-adic norms

(The F.T.A.) Every $n \geqslant 1$ admits a **unique prime decomposition** :

$$n = 2^{x_0} \times 3^{x_1} \times 5^{x_2} \times 7^{x_3} \times 11^{x_4} \times \ldots$$

where a finite number of these $\{x_j\}_{j \in \mathbb{N}}$ are non-zero. Think of these as **registers**.

Each fraction becomes a *conditional* **increment** and **branch**

$$\frac{5^3 \times 11^2}{3^3 \times 7^1} \qquad \mathrm{COND}\ (x_1 \geqslant 3 \ \mathrm{AND}\ \ x_3 \geqslant 1) \qquad \begin{array}{l}\{x_2 \mapsto x_2 + 3; \\ \ x_4 \mapsto x_4 + 2; \\ \ \mathrm{BRANCH}; \}\end{array}$$

Conditionals 'consume resources' :

Each (successful) conditional $(x_j \geqslant M)$ decrements the **register** by the **test value** $x_j \mapsto x_j - M$.

Taking reciprocals interchanges conditionals/decrements with instructions/increments.

Motivation?

FRACTRAN came out of Conway's work[2] on undecidability in elementary arithmetic

"Unpredictable Iterations" – J. H. Conway (1972)

This was demonstrated via the halting problem, based on

computational universality of iterative problems on **congruential functions**.

Functions defined "piece-wise linearly on modulo classes"

A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is **congruential** when there exists a set of modulo classes $\{A_j\mathbb{N} + B_j\}$ such that

$$f(n) = \frac{X_i n + Y_i}{Z_i} \quad \text{where} \quad n \equiv B_i \mod A_i$$

[2] See also "On E. L. Post's Tag problem" – S. Maslov (1964)

Some non-trivial motivating examples

1. The **Notorious Collatz Conjecture**: $n \mapsto \begin{cases} \frac{n}{2} & n \text{ even}, \\ \frac{3n+1}{2} & n \text{ odd}. \end{cases}$

   (Conjecture: every iterated sequence eventually arrives at 1).

2. The **Original Collatz Conjecture** (L. C., unpublished notebooks, 1 July, 1932, as described by J. Lagarias (1985)).

J. Conway's '*Amusical Permutation*'

$$\text{The congruential bijection } \gamma(n) = \begin{cases} \frac{2n}{3} & n \equiv 0 \mod 3, \\ \frac{4n-1}{3} & n \equiv 1 \mod 3, \\ \frac{4n+1}{3} & n \equiv 2 \mod 3, \end{cases}$$

(Conjecture: This function has infinite orbits – the orbit of 8 is ∞).

Conway described 2. (the OCC) as,

> *"The best candidate we have for a* **true** *but* **unprovable** *statement."*

Conjectures as code??

Both the N.C.C. and the O.C.C. are undecided – possibly undecidable[3].

Can we nevertheless *understand* them better,

by interpreting them as `FRACTRAN` programs?

A correspondence of Conway:

Every `FRACTRAN` program implements a **purely multiplicative** congruential function

$$f(n) = \frac{X_i}{Y_i} n + 0 \qquad n \equiv B_i \mod A_i$$

. . . rather neatly ruling out his motivating examples.

A question: Instead of looking at simple procedural languages, should we interpret congruential functions via (categorical) logic / lambda calculus??

[3] Although we could never prove undecidability(!)

The domain of congruential functions

Congruential functions are defined piece-wise linearly on *dissections of* $\mathbb{N}$ *into disjoint modulo classes*.

Exact Covering Systems (P. Erdös et al., 1950s - present)

An E.C.S. is an indexed family of modulo classes $\mathcal{C} = \{A_j\mathbb{N} + B_j\}_{j\in J}$ where:

- $\mathbb{N} = \bigcup_{j\in J} A_j\mathbb{N} + B_j$
- $(A_i\mathbb{N} + B_i) \cap (C_j\mathbb{N} + D_j) = \emptyset \quad \forall\, i \neq j$

Instead of *indexed sets*, we should take a more *dynamical, algebraic* view, and define them via arrows in a category (monoid).

A complete triviality Each modulo class $A_j\mathbb{N} + B_j$ of an e.c.s. is countably infinite, and so in 1:1 correspondence with $\mathbb{N}$ itself.

Which arrows exhibit these correspondences,
& what properties must they satisfy?

Dynamical Algebras via exact covering systems (I)

Let $\mathcal{C} = \{A_i\mathbb{N} + B_i\}_{i=1}^n$ be an exact covering system

1. $A_i\mathbb{N} + B_i$ **is bijective with** $\mathbb{N}$ for all $i = 1..n$

   This is exhibited by the injection : $p_i(n) \stackrel{def.}{=} A_i n + B_i$

2. **Elements of C are pairwise disjoint**

   becomes the condition : $im(p_i) \cap im(p_j) = \varnothing \quad \forall\, i \neq j$

3. **Elements of $\mathcal{C}$ cover the whole of** $\mathbb{N}$

   is equivalent to : $\bigcup_{i=1}^{n} im(p_i) = \mathbb{N}$

Instead of discussing *domains* and *images* . . .

We work with the 'symmetric inverse monoid' $\mathcal{I}(\mathbb{N})$ of **partial injective functions**

$$f(x) = f(x') \ \Rightarrow \ x = x' \quad \text{provided both are defined.}$$

and their unique **generalised inverses** $f(x) = y \ \Leftrightarrow \ f^*(y) = x$

The generalised inverse of p_j is given by

$$p_j^*(n) = \begin{cases} \frac{n-B_j}{A_j} & n \equiv B_j \mod A_j \\ \perp & \text{otherwise} \end{cases}$$

The required conditions are the defining relations of the n-th **dynamical algebra** :

$$p_j^* p_i = \begin{cases} Id & i = j \\ \varnothing & i \neq j \end{cases} \quad \text{and} \quad \sum_{i=1}^{n} p_i p_i^* = Id$$

where 'sum' is union of partial functions with disjoint domains / images.

(Categorically, "sum" is the join w.r.t. the poset-enrichment of the category of partial injective functions).

Congruential functions in logic & lambda calculii

Given an exact covering system $\{A_i\mathbb{N} + B_i\}_{i=0}^{n-1}$, the corresponding partial injections

$$p_0\ ,\ p_1\ ,\ \ldots\ ,\ p_{n-1}\ :\ \mathbb{N} \rightarrow \mathbb{N}$$

generate a copy of the n-th **dynamical algebra** (a.k.a. Nivat & Perot's n-th polycyclic monoid), as (partial) congruential functions.

These appear in a series of (closely related) logical / computational models :

M.E.L.L. and **system** $\mathcal{F}$ *Geometry of Interaction I and II*
– J.-Y. Girard (1988,1990),

untyped lambda calculus *Local and asynchronous* β*-reduction*
– V. Danos & L. Regnier (1992),

combinatory logic *GoI and linear combinatory algebra*
– S. Abramsky, E. Hagverdi, P. Scott (2002),

. . .

The examples that motivated Conway arise when
we consider the **category theory** behind these models.

A categorically significant example

In J.-Y. Girard's Geometry of Interaction system (Parts I, II) :

Propositions are modelled by partial injective functions – elements of $\mathcal{I}(\mathbb{N})$.

Conjunction is defined in terms of the two-generator dynamical algebra, based on the odd-even dissection $\mathbb{N} = 2\mathbb{N} \cup 2\mathbb{N}+1$.

Expanding out his definition, we get :

$$(f \star g)(2n) = 2.f(n)$$

$$(f \star g)(2n+1) = 2.g(n)+1$$

A simple description, based on Hilbert's Grand Hotel

This "writes two functions as a single function", by

replicating their behaviour on the **even** *and* **odd** *numbers respectively.*

Key point : Girard's conjunction $(\_ \star \_) : \mathcal{I}(\mathbb{N}) \times \mathcal{I}(\mathbb{N}) \hookrightarrow \mathcal{I}(\mathbb{N})$ is a (semi-monoidal) categorical tensor on a monoid.

Congruential Canonical Isomorphisms

Girard's conjunction cannot be strictly associative

Strict associativity of a faithful (injective) tensor on a monoid

$$\Longleftrightarrow$$

The unique object of the monoid is the unit object.

Coherence & Strictification for Self-Similarity
Journal Homotopy & Related Structures *(P.M.H. 2016)*

There is a non-trivial natural isomorphism $(\_ \star (\_ \star \_)) \Rightarrow ((\_ \star \_) \star \_)$

$$\alpha(a \star (b \star c)) \;=\; ((a \star b) \star c)\alpha \quad \forall\, a, b, c \in \mathcal{I}(\mathbb{N})$$

whose unique component (the **associator**) $\alpha(n) \;=\; \begin{cases} 2n & n \equiv 0 \mod 2, \\ n+1 & n \equiv 1 \mod 4, \\ \frac{n-1}{2} & n \equiv 3 \mod 4, \end{cases}$

is a congruential function, satisfying MacLane's pentagon condition

$$\alpha^2 \;=\; (\alpha \star Id)\, \alpha\, (Id \star \alpha)$$

Collatz-like problems for canonical isomorphisms?

Question :

Can we decide whether orbits of natural numbers under the associativity isomorphism α are **finite** or **infinite**?

The associator α has disappointingly simple behaviour under iteration :

- n is even: $\alpha(n) = 2n > n$ for all $n \neq 0$
- n is odd: Two possibilities :
  1. $\alpha(n)$ is odd $\Rightarrow \alpha(n) < n$
  2. $\alpha(n)$ is even.

All orbits, apart from the fixed point $\alpha(0) = 0$, are infinite.

We see similar *disturbingly trivial* behaviour under iteration for arbitrary associativity isomorphisms[4]

[4] i.e. components of natural isomorphisms between k-fold bracketings of Girard's conjunction.

Briefing for a descent into group theory

The structure group for the associativity identity, P. Dehornoy (1996)

Given a tensor on a monoid, the associativity isomorphisms form a group, isomorphic to R. Thompson's group $\mathcal{F} = \langle X_j \ : \ X_j X_k X_j^{-1} = X_{k-1} \quad \forall \, j < k-1 \in \mathbb{N} \rangle$.

A suitable generating set is given by

$$X_0 = \alpha \ , \ X_1 = (Id \star \alpha) \ , \ X_2 = Id \star (Id \star \alpha) \ , \ \ldots$$

since

$$\alpha(Id \star (Id \star X_j))\alpha^{-1} = (Id \star Id) \star \alpha = Id \star \alpha$$

$X_0, X_1, X_2, \ldots$ can be thought of as: "α replicated on

$$2\mathbb{N}+1 \ , \ 4\mathbb{N}+3 \ , \ 8\mathbb{N}+7 \ , \ , \ldots$$

and the identity elsewhere."

Again, all orbits are either **infinite** or **fixed points**.

Characterising canonical isomorphisms for Girard's conjunction

A standard result

The above 'infinite presentation' of Thompson's $\mathcal{F}$ is not minimal;

X_0 and X_1 suffice to generate the whole group.

Corollary The two bijections

$$\alpha(n) = \begin{cases} 2n & n \ (mod\ 2) = 0 \\ n+1 & n \ (mod\ 4) = 1 \\ \frac{n-1}{2} & n \ (mod\ 4) = 3 \end{cases} \qquad (Id \star \alpha)(n) = \begin{cases} n & n \ (mod\ 2) = 0 \\ 2n-1 & n \ (mod\ 4) = 1 \\ n+2 & n \ (mod\ 8) = 3 \\ \frac{n-1}{2} & n \ (mod\ 8) = 7 \end{cases}$$

generate a copy of Thompson's group $\mathcal{F}$, and so capture *all* canonical associativity isomorphisms for $\_\star\_$.

None of these have interesting behaviour under iteration!

An important question(!)

Are congruential canonical isomorphisms always so uninteresting[5]??

"Given any n objects of a monoidal category, the canonical associativity isomorphisms give a commuting diagram whose shape is the 1-skeleton of the n^{th} associahedron $\mathcal{K}_n$". — M. Kapranov (1993)

The claim :

We see **non-trivial behaviour**[a] when we move off the 1-skeleton, and consider commuting diagrams of maps between more general facets of associahedra.

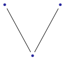
[a]Precisely, the operators of Collatz that motivated Conway ...

[5]from an *iterative*, rather than *group-theoretic* viewpoint!

An unbiased series of conjunctions

Girard gave a **binary** model of conjunction $(\_ \star \_) : \mathcal{I}(\mathbb{N}) \times \mathcal{I}(\mathbb{N}) \hookrightarrow \mathcal{I}(\mathbb{N})$.

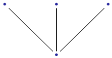
"$(a \star b)$ replicates a, b on the congruence classes $2\mathbb{N}$, $2\mathbb{N}+1$ respectively".

- We draw this as 

There is an obvious **ternary** analogue, $(\_ \star \_ \star \_) : \mathcal{I}(\mathbb{N}) \times \mathcal{I}(\mathbb{N}) \times \mathcal{I}(\mathbb{N}) \hookrightarrow \mathcal{I}(\mathbb{N})$

$$(a \star b \star c)(3n+i) \ = \ \begin{cases} 3.a(n) & i = 0 \\ 3.b(n)+1 & i = 1 \\ 3.c(n)+2 & i = 2 \end{cases}$$

"Replicate a, b, c on the congruence classes $3\mathbb{N}$, $3\mathbb{N}+1$, $3\mathbb{N}+2$ respectively".

- We draw this as 

The general case :

For any $k \geqslant 1$, we form the k^{th} **unbiased conjunction** by :

$$(f_0 \star \ldots \star f_{k-1})(kn+i) \;=\; k.f_i(n)+i \;\text{ where } i = 0,1,2,\ldots,k-1$$

Alternatively & equivalently,

$$(f_0 \star \ldots f_{k-1})(x) \;=\; k.f_i\left(\frac{x-i}{k}\right)+i \;\text{ where } x \equiv i \mod k$$

This gives, for any $k > 0$, an injective homomorphism $\mathcal{I}(\mathbb{N})^{\times k} \hookrightarrow \mathcal{I}(\mathbb{N})$ that :

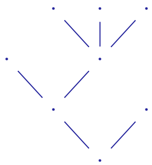
replicates f_0 , f_1 , $\ldots$, f_{k-1} *on the congruence classes modulo* k

i.e. $\{k\mathbb{N} \; , \; k\mathbb{N}+1 \; , \; \ldots \; , \; k\mathbb{N}+(k-1)\}$

For $k = 1,2,3,4,\ldots$, we draw the unbiased conjunctions as

We may compose these, to build an **operad**.

These 'compose by substitution' to give an operad $\mathcal{GC}$ of **generalised conjunctions**.
Each k-leaf tree determines an injective hom. $\mathcal{I}(\mathbb{N})^{\times k} \hookrightarrow \mathcal{I}(\mathbb{N})$.



$$: \ \mathcal{I}(\mathbb{N})^{\times 5} \hookrightarrow \mathcal{I}(\mathbb{N})$$

$$(f_0, f_1, f_2, f_3, f_4) \quad \mapsto \quad ((f_0 \star (f_1 \star f_2 \star f_3)) \star f_4)$$

More formally :

Consider the (non-symmetric) endomorphism operad of $\mathcal{I}(\mathbb{N})$ within the category $(\mathbf{Inv}, \times)$ of inverse monoids, with Cartesian product. This contains one *unbiased conjunction* of each arity > 0.

These generate the sub-operad $\mathcal{GC}$ of generalised conjunctions.

A freely generated operad

Claim : The operad $\mathcal{GC}$ is isomorphic to the operad $\mathbb{RPT}$ of **rooted planar trees**.

Thus

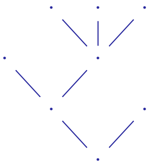
- It is *freely* generated by the unbiased conjunctions.
- Every distinct rooted planar tree determines a distinct generalised conjunction.
- The n-ary operations $\mathcal{GC}_n$ are in 1:1 correspondence with facets of the n^{th} associahedron $\mathcal{K}_n$.

To be proved :

Each tree in $\mathbb{RPT}_k$ determines a *distinct* homomorphism $\mathcal{I}(\mathbb{N})^{\times k} \hookrightarrow \mathcal{I}(\mathbb{N})$.

Back to simple arithmetic!

The tree



defines a homomorphism : $\mathcal{I}(\mathbb{N})^{\times 5} \hookrightarrow \mathcal{I}(\mathbb{N})$

In the generalised conjunction $((f_0 \star (f_1 \star f_2 \star f_3)) \star f_4)$, the action of each f_j is mapped :

from The whole of the natural numbers $\mathbb{N}$

to Some modulo class $A_j\mathbb{N} + B_j$.

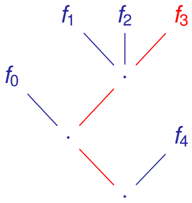
For example : f_3 is 'replicated' on the modulo class $12\mathbb{N} + 10$.

Obvious Question:

How do we derive these coefficients *from the tree*?

A root and branch approach

Deriving $12\mathbb{N} + 10$, from the leaf-to-root path :



Branch number 2 of 3

Branch number 1 of 2

Branch number 0 of 2

Multiplicative coefficient : $12 = 3 \times 2 \times 2$

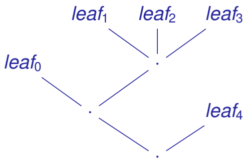
Additive coefficient :	(Decimal)		Base 3	Base 2	Base 2
	10	=	2	1	0

Positional mixed-radix number systems

First formal study by G. Cantor, *Über einfache Zahlensysteme* (1869)

Covering the Numbers with Trees

Rooted Planar Trees are uniquely determined by the addresses of their leaves



leaf$_0$		$(0,2)$	$(0,2)$	$4\mathbb{N}$
leaf$_1$	$(0,3)$	$(1,2)$	$(0,2)$	$12\mathbb{N}+2$
leaf$_2$	$(1,3)$	$(1,2)$	$(0,2)$	$12\mathbb{N}+6$
leaf$_3$	$(2,3)$	$(1,2)$	$(0,2)$	$12\mathbb{N}+10$
leaf$_4$			$(1,2)$	$2\mathbb{N}+1$

which uniquely determine ordered **exact covering systems**, such as

$$4\mathbb{N}\ ,\ 12\mathbb{N}+2\ ,\ 12\mathbb{N}+6\ ,\ 12\mathbb{N}+10\ ,\ 2\mathbb{N}+1$$

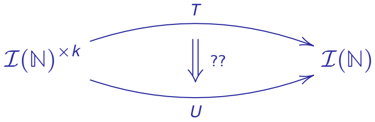
Corollaries:

1. Distinct trees determine distinct homomorphisms, so $\mathcal{GC} \cong \mathbb{RPT}$.
2. Distinct k-leaf trees determine distinct realisations of the k-th dynamical algebra.

Mapping between generalised conjunctions

The operations of $\mathcal{GC}_k$ are homomorphisms (functors).

Given generalised conjunctions $T, U : \mathcal{I}(\mathbb{N})^{\times k} \hookrightarrow \mathcal{I}(\mathbb{N})$, how can we find (well-behaved) natural isomorphisms between them?

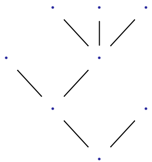


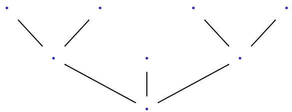
Convention : As generalised conjunctions are **monoid** homomorphisms, natural transformations have a **single** component.

We identify nat. iso.s with their unique component in $\mathcal{I}(\mathbb{N})$.

Congruential functions as natural isomorphisms

Consider the generalised conjunctions[6] $T, U : \mathcal{I}(\mathbb{N})^{\times 5} \hookrightarrow \mathcal{I}(\mathbb{N})$

$T =$ 

$U =$ 

We build a natural isomorphism $\eta_{T,U} : T \Rightarrow U$ by monotonically mapping between their respective ordered exact covering systems :

leaf 0	$4\mathbb{N}$	$\mapsto$	$6\mathbb{N}$
leaf 1	$12\mathbb{N}+2$	$\mapsto$	$6\mathbb{N}+3$
leaf 2	$12\mathbb{N}+6$	$\mapsto$	$3\mathbb{N}+1$
leaf 3	$12\mathbb{N}+10$	$\mapsto$	$6\mathbb{N}+2$
leaf 4	$2\mathbb{N}+1$	$\mapsto$	$6\mathbb{N}+4$

This gives, as desired,

$$\eta_{T,U}.((a \star (b \star c \star d)) \star e) \;=\; ((a \star b) \star c \star (d \star e)).\eta_{T,U}$$

[6] edges of the fifth associahedron $\mathcal{K}_5$

Concrete formulæ for congruential natural isomorphisms

Given two ordered exact covering systems, determined by k-ary generalised conjunctions T, U

$$\begin{array}{llcl}
\textbf{leaf } 0 & A_0\mathbb{N} + B_0 & \mapsto & C_0\mathbb{N} + D_0 \\
\textbf{leaf } 1 & A_1\mathbb{N} + B_1 & \mapsto & C_1\mathbb{N} + D_1 \\
\vdots & \vdots & & \vdots \\
\textbf{leaf } k-1 & A_{k-1}\mathbb{N} + B_{k-1} & \mapsto & C_{k-1}\mathbb{N} + D_{k-1}
\end{array}$$

The natural isomorphism $\eta_{T,U}$ is the congruential bijection

$$\eta_{T,U}(x) = \frac{1}{A_j}\left(C_j x + \begin{vmatrix} A_j & B_j \\ C_j & D_j \end{vmatrix}\right) \text{ where } x \equiv B_j \mod A_j$$

Think of these as, "mapping between distinct realisations of the k-th dynamical algebra".

Some context :

These are a *special kind* of congruential function, where

each element of an e.c.s. is mapped linearly to another modulo class.

- **A computational interpretation :**

  Functional equations associated with congruential functions,

  Theoretical Computer Science *S. Berckel (1994)*

- **Pure group theory :**

  A simple group given by interchanging residue classes of the integers

  Mathematische Zeitschrift *S. Kohl (2010)*

- **Close connections with the N.C.C. :**

  The [$3x + 1$] Collatz conjecture in a group-theoretic context

  Journal of Group Theory *S. Kohl (2017)*

- **Normal forms & formulæ for composition :**

  The inverse semigroup theory of elementary arithmetic

  www.arxiv.org/2206.07412 *P.M.H. (2022)*

A groupoid of generalised conjunctions

Observe that :

- $\eta_{T,T} = Id \in \mathcal{I}(\mathbb{N})$
- $\eta_{T,U}\eta_{S,T} = \eta_{S,U}$
- $\eta^{-1}_{T,U} = \eta_{U,T}$

We have a **posetal groupoid**[7] $\mathcal{C}\omega$ of functors / natural iso.s, given by :

Objects Generalised conjunctions (operations of $\mathcal{GC}$)

Arrows $\mathcal{C}\omega(T,U) = \begin{cases} \{\eta_{T,U}\} & T, U \text{ have the same arity,} \\ \varnothing & \text{otherwise.} \end{cases}$

A reminder :

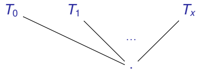
Arrows are natural isomorphisms between homomorphisms $\mathcal{I}(\mathbb{N})^{\times k} \hookrightarrow \mathcal{I}(\mathbb{N})$ determined by their unique components in $\mathcal{I}(\mathbb{N})$

[7] within which, 'all diagrams commute'

Claim: As well as being the **objects** of the posetal groupoid $\mathcal{C}\omega$, generalised conjunctions are also **functors** $\prod^k \mathcal{C}\omega \rightarrow \mathcal{C}\omega$.

On Objects : this is defined by operadic composition

$$(T_0 \star \ldots \star T_x) \overset{def.}{=}$$



On Arrows : We take the generalised conjunction of unique components.

To check **functoriality :**

$$\eta_{(T_0 \star \ldots \star T_x),(U_0 \star \ldots \star U_x)} = (\eta_{T_0,U_0} \star \ldots \star \eta_{T_x,U_x})$$

$$\begin{array}{cccc} T_0 & T_1 & \ldots & T_k \\ \eta_{T_0,U_0} \Downarrow & \eta_{T_1,U_1} \Downarrow & & \eta_{T_0,U_0} \Downarrow \\ U_0 & U_1 & \ldots & U_x \end{array}$$

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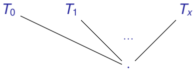

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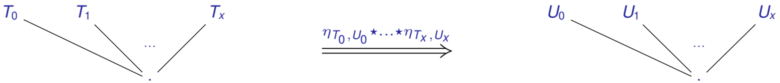


Operads all the way down

Key points :

- The groupoid $\mathcal{C}\omega$ is **posetal**, with objects in 1:1 correspondence with facets of associahedra.
- There is a **unique arrow** between any two facets of the same associahedron.
- It admits an $\mathbb{N}^+$-indexed family of faithful functors :

$$\begin{array}{rcl} Id & : & \mathcal{C}\omega \rightarrow \mathcal{C}\omega \\ (\_ \star \_) & : & \mathcal{C}\omega \times \mathcal{C}\omega \rightarrow \mathcal{C}\omega \\ (\_ \star \_ \star \_) & : & \mathcal{C}\omega \times \mathcal{C}\omega \times \mathcal{C}\omega \rightarrow \mathcal{C}\omega \\ & \vdots & \end{array}$$

- As a **very special case**, it contains a copy of MacLane's posetal monoidal groupoid $(\mathcal{W}, \_\square\_)$.

Fun question :

What does the endomorphism operad[a] of $\mathcal{C}\omega$ look like, in the category $(\mathbf{Cat}, \times)$?

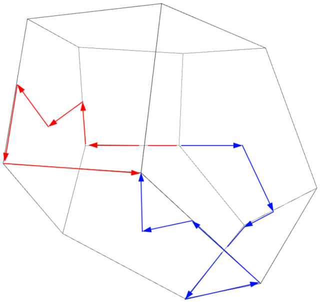
[a] or functor category thereof . . .

Commuting diagrams from associahedra

As $\mathcal{C}\omega$ is posetal, all diagrams over it commute.

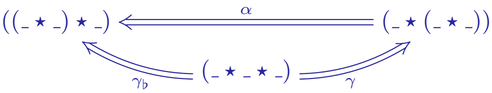
There exists a unique arrow between any two objects
iff
they are facets of the same associahedron.

We may build commuting diagrams from paths between arbitrary facets.



Labels on edges are **congruential functions**.

Concrete examples : the third associahedron



$$\alpha(n) = \begin{cases} 2n & n \equiv 0 \mod 2, \\ n+1 & n \equiv 1 \mod 4, \\ \frac{n-1}{2} & n \equiv 3 \mod 4. \end{cases} \qquad \gamma(n) = \begin{cases} \frac{2n}{3} & n \equiv 0 \mod 3, \\ \frac{4n-1}{3} & n \equiv 1 \mod 3, \\ \frac{4n+1}{3} & n \equiv 2 \mod 3. \end{cases}$$

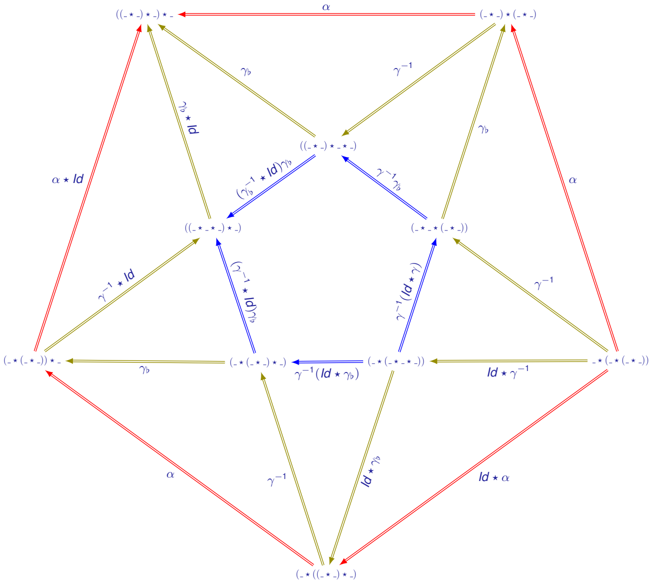
$$\gamma_\flat(n) = \begin{cases} \frac{4n}{3} & n \equiv 0 \mod 3, \\ \frac{4n+2}{3} & n \equiv 1 \mod 3, \\ \frac{2n-1}{3} & n \equiv 2 \mod 3. \end{cases}$$

We have already seen these (!)

- α : the **associativity isomorphism** for Girard's conjunction.
- γ : the '**amusical permutation**', from the Original Collatz Conjecture.
- $\gamma_\flat$ related to γ by a natural transformation : $1 + \gamma_\flat(n) = \gamma(n+1)$.

  This *'replicates the behaviour of γ, one step down'*.

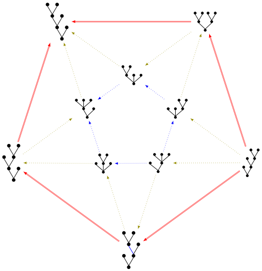
From the third to the fourth associahedron



A Convergent Series?

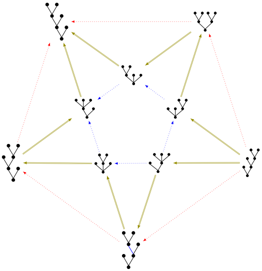
MacLane's pentagon is the 1-skeleton of Stasheff's associahedron $\mathcal{K}_4$;
we understand the rest of the pentagram in similar terms.

Mapping between Vertices



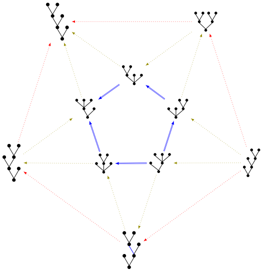
The associator for Girard's $(\_ \star \_)$

Mapping between Edges & Vertices



Collatz's bijection(s)

Mapping between Edges



Girard-Collatz composites

These are natural isomorphisms in a posetal functor category
and also, congruential bijections on the natural numbers.

One of Conway's motivating examples (the O.C.C.) arises from :

- Coherence for **associativity**
- Re-bracketing generally

The other motivating example (the N.C.C.) is seen when we extend to :

- Coherence for **symmetry & associativity**,
- Non-symmetric operads.

The core operator from the $3x+1$ problem is not a bijection!

The problem has, however, been characterised in terms of (congruential) bijections

Claim : these are coherence isomorphisms

A relevant reference(!)

Journal of Group Theory, Stefan Kohl (2017)

"The *[Notorious]* Collatz conjecture in a group-theoretic context"

Many thanks to Matt Brin (Binghampton) for bringing this to my attention!

Introduces a three-generator group of permutations $\mathcal{G}_{\mathcal{T}} \subseteq \mathcal{I}(\mathbb{N})$ satisfying :

"$\mathcal{G}_{\mathcal{T}}$ acts transitively[8] on $\mathbb{N}$ iff the $3x + 1$ conjecture is true".

The generators – and hence all elements of $\mathcal{G}_{\mathcal{T}}$ – are **congruential bijections**.

Claim : Once we include symmetry, these are natural isomorphisms between generalised conjunctions, describing coherence for $(\_ \star \_)$ and $(\_ \star \_ \star \_)$.

[8] For all $x, y \in \mathbb{N}$, there exists $f \in \mathcal{G}_{\mathcal{T}}$ such that $f(x) = y$.

What has been missing . . .

Girard's conjunction $\_ \star \_ : \mathcal{I}(\mathbb{N}) \times \mathcal{I}(\mathbb{N}) \rightarrow \mathcal{I}(\mathbb{N})$ is a **symmetric** tensor.

$$\sigma(f \star g) \ = \ (g \star f)\sigma \quad \forall\, g, f \in \mathcal{I}(\mathbb{N})$$

where the **symmetry iso.** $\sigma \in \mathcal{I}(\mathbb{N})$ is given by : $\sigma(n) \ = \ \begin{cases} n+1 & n \text{ even,} \\ n-1 & n \text{ odd.} \end{cases}$

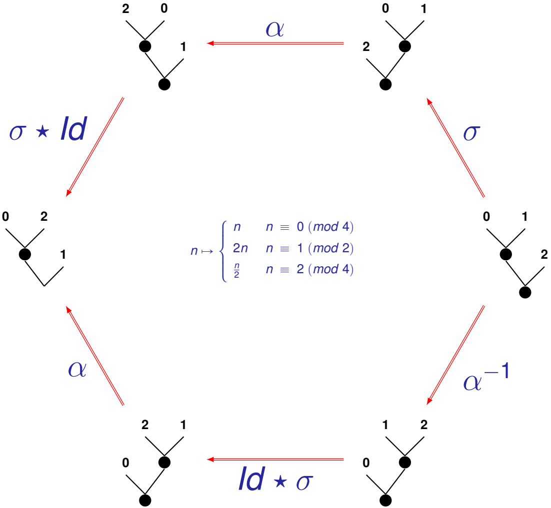
Coherence for symmetry requires **MacLane's hexagon condition** :

$$\alpha\sigma\alpha \ = \ (\sigma \star Id)\alpha(Id \star \sigma)$$

which arises from the 1-skeleton of Kapranov's third **permutoassociahedron**, $\mathcal{KP}_3$, based on *symmetric operads* with an action of $\mathcal{S}(\{0, \ldots, k-1\})$ on leaves of trees.

MacLane's hexagon for Girard's conjunction

Vertices are **bracketings** of **orderings** of three symbols, $\{0, 1, 2\}$.



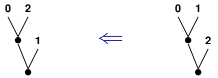
$$n \mapsto \begin{cases} n & n \equiv 0 \ (mod\ 4) \\ 2n & n \equiv 1 \ (mod\ 2) \\ \frac{n}{2} & n \equiv 2 \ (mod\ 4) \end{cases}$$

The generators of $\mathcal{G}_{\mathcal{T}}$

From $\mathcal{KP}_2$ The symmetry isomorphism for Girard's conjunction, $\sigma \in \mathcal{I}(\mathbb{N})$

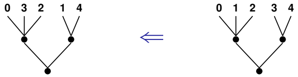
$$n \mapsto \begin{cases} n+1 & n \equiv 0 \ (mod\ 2) \\ n-1 & n \equiv 1 \ (mod\ 2) \end{cases}$$

From $\mathcal{KP}_3$ The bijection from MacLane's hexagon $(\sigma \star Id)\alpha\sigma = \alpha(Id \star \sigma)\alpha^{-1}$



$$n \mapsto \begin{cases} n & n \equiv 0 \ (mod\ 4) \\ 2n & n \equiv 1 \ (mod\ 2) \\ \frac{n}{2} & n \equiv 2 \ (mod\ 4) \end{cases}$$

From $\mathcal{KP}_5$ A mapping between *edges* of the fifth[9] permutoassociahedron.



$$n \mapsto \begin{cases} n & n \equiv 0 \ (mod\ 6) \\ \frac{2n+2}{3} & n \equiv 2 \ (mod\ 6) \\ n & n \equiv 4 \ (mod\ 6) \\ \frac{3n-1}{2} & n \equiv 1 \ (mod\ 4) \\ n & n \equiv 2 \ (mod\ 4) \end{cases}$$

All of these decompose into composites *&* generalised conjunctions *of* $\alpha, \sigma, \gamma, \gamma_\flat$.

[9] Caution : KP_n is not **simple**, for $n \geqslant 4$.

Using vertex-to-vertex and vertex-to-edge maps, we implement the required edge-to-edge map.