Consistent ultrafinitist logic

CLA2023 presentation

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  - Transfinitist mathematics

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  - redefining decidability
  - philosophy of inference

Background

Transfinitist mathematics

Assumption of infinity:

- Numbers
- Enumerations

Finitism

- Finite proofs

Finitism

- Finite proofs
- Finite descriptions of proofs

Finitism

- Finite proofs
- Finite descriptions of proofs
- Examination of infinite proofs is unfeasible

Physical limits

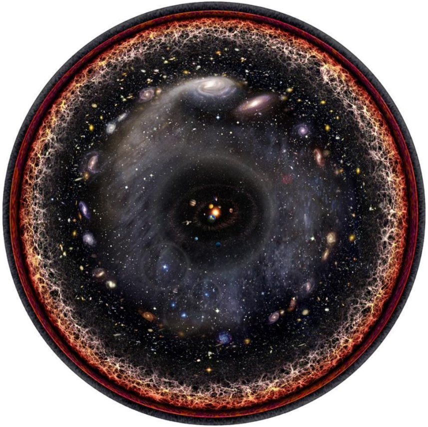

Figure 1: Observable universe in log scale

Ultrafinitism

- Feasible numbers (Sazonov 1995)

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- Limit of observable universe (Lloyd 2002)

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- Physical limits of accuracy (Gisin 2019)
- Limit of computational density (Krauss and Starkman 2004)
- Gorelik-Bremermann limit (Gorelik 2010)

Bumpy road to ultrafinitism

- “no satisfactory developments exist” (Troelstra 1988)

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Bumpy road to ultrafinitism

- “no satisfactory developments exist” (Troelstra 1988)
- Bounded Arithmetic (Krajicek 1995)
- Primitive Recursive Functions are not all finitist functions (Schirn and Niebergall 2005)
- naive finitism logic as inconsistent (Dummett 1975; Magidor 2007)

Proposed: ultraconstructivism

- consider only constructive developments

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- carry explicit computational bounds

Proposed: ultraconstructivism

- consider only constructive developments
- carry explicit computational bounds
- check that proof exists within those bounds

Consistent Ultrafinitist Logic

Syntax

Sizes

$$\begin{array}{lrcl} \text{Size variables:} & v & \in & V \\ \text{Positive naturals:} & i & \in & \mathbb{N} \setminus \{0\} \end{array}$$

Cost polynomials

$$n \geq 1$$

$$\begin{array}{lcll} \text{Polynomials:} & \rho & ::= & v \mid i \mid \rho + \rho \mid \rho * \rho \mid \rho^{\rho} \mid \mathit{iter}(\rho, \rho, v) \mid \rho[\![{}^{v}/_{\rho}]\!] \\ \text{Data size bounds:} & \alpha & ::= & \rho \\ \text{Computation bounds:} & \beta & ::= & \rho \end{array}$$

Iterated function composition: $\mathit{iter}(\rho_1, \rho_2, v)$ Function ρ_1 with a bound variable v is iterated ρ_2 times.

Types

$$
\begin{array}{lcl}
\text{Type or term variables:} & x & \in & X \\
\text{Types:} & \tau & ::= & v \big| \tau \wedge \tau \big| \tau \vee \tau \big| \forall x_v : \tau \rightarrow^{\alpha}_{\beta} \tau \big| \bot \big| \circ \\
\text{Environments:} & \Gamma & ::= & v_1 : {\tau^1}_{\beta_1}, ..., {\tau^n}_{\beta_n} \\
\text{Judgements:} & J & ::= & \Gamma \vdash^{\alpha}_{\beta} E : \tau
\end{array}
$$

x_v - term variable x with its size bound by size variable v

Notation $\forall x_v : A \implies^{\alpha(v)}_{\beta(v)} B$ binds proof variable x with type of A, and then bound in polynomials $\alpha(v)$ for complexity and $\beta(v)$ for depth of the normalized term.

Proof terms

$$\text{Terms:} \quad E \quad ::= \quad v \big| \lambda v.E \big| in_r(E) \big| in_l(E) \big| (E, E) \big| \, () \\ \quad \big| \quad case\,E\,of \begin{array}{lcl} in_l(v) & \rightarrow & E; \\ in_r(v) & \rightarrow & E; \end{array}$$

β_i is an upper bound on depth of term proving A^i.

Judgements

Type or term variables:	x	$\in$	X
Types:	τ	$::=$	$v \| \tau \wedge \tau \| \tau \vee \tau \| \forall x_v : \tau \rightarrow^{\alpha}_{\beta} \tau \| \bot \| \circ$
Environments:	Γ	$::=$	$v_1 : {\tau^1}_{\beta_1}, ..., {\tau^n}_{\beta_n}$
Judgements:	J	$::=$	$\Gamma \vdash^{\alpha}_{\beta} E : \tau$

- x_v - variable its size bound variable
- α_i is an upper bound on computation steps needed to evaluate A^i.
- β_i is an upper bound on depth of term proving A^i.

Rules

Variables

$$\frac{\Gamma \vdash^{\alpha}_{\beta} y_{\beta} : A \quad v \in V}{\Gamma, x_{\beta} : A \vdash^{1}_{\beta} x : A} \; var$$

Unit type

$$\frac{}{\Gamma \vdash^{1}_{\beta} () : \circ}\ \textit{unit}$$

Subsumption

$$\frac{\Gamma \vdash^{\alpha_1}_{\beta_1} e : A \qquad \alpha_1 \leq \alpha_2 \qquad \beta_1 \leq \beta_2}{\Gamma \vdash^{\alpha_2}_{\beta_2} e : A}\ \textit{subsume}$$

Positive polynomials for easy bounds

Positive polynomials for easy bounds

$$x, y, e, f, g, h \geq 1$$

$$a, b \geq 0$$

$$\begin{array}{lrcl}
(1) & a * x^e + b * x^f & \leq & (a + b) * x^f \\
(2) & a * x^e * y^g & \leq & a * x^f * y^h \\
(3) & iter(e, g, x) & \leq & iter(f, h, x) \\
(4) & iter(a * x, e, x) & = & a^e * x \\
(5) & iter(x + a, e, x) & = & x + a * e \\
(6) & iter(x^e, g, x) & = & x^{e^g}
\end{array}$$

Conjunction

$$\frac{\Gamma \vdash^{\alpha_1}_{\beta_1} a^1 : A^1 \quad \Gamma \vdash^{\alpha_2}_{\beta_2} a^2 : A^2}{\Gamma \vdash^{\alpha_1+\alpha_2}_{\max(\beta_1,\beta_2)+1} (a^1, a^2) : A^1 \wedge A^2} \; pair$$

$$\frac{\Gamma \vdash^{\alpha}_{\beta+1} e : A^1 \wedge A^2 \quad i \in \{1, 2\}}{\Gamma \vdash^{\alpha+1}_{\beta} prj_i \; e : A^i} \; prj_i$$

Alternative

$$\frac{\Gamma \vdash^{\alpha}_{\beta} e : A^i \quad i \in \{l, r\}}{\Gamma \vdash^{\alpha+1}_{\beta+1} in_i(e) : A^1 \vee A^2} \ inj$$

$$\frac{\Gamma \vdash^{\alpha_\vee}_{\beta_\vee+1} a : A^1 \vee A^2 \qquad \Gamma, x : {A^1}_{\beta_\vee} \vdash^{\alpha_1}_{\beta_1} b : B \qquad \Gamma, y : {A^2}_{\beta_\vee} \vdash^{\alpha_2}_{\beta_2} c : B}{\Gamma \vdash^{\alpha_\vee + max(\alpha_1,\alpha_2)+1}_{\max(\beta_1,\beta_2)} case\ a\ of \begin{array}{lcl} in_l(x) & \to & b; \\ in_r(y) & \to & c; \end{array} : B} \ case$$

Abstraction and application

$$\frac{\Gamma, x_v : A \vdash^{\alpha(v)}_{\beta(v)} e : B}{\Gamma \vdash^{\alpha(1)+1}_{\beta(1)+1} \lambda x.e : \forall a_v : A \rightarrow^{\alpha(v)}_{\beta(v)} B} \; abs$$

$$\frac{\Gamma \vdash^{\alpha_1}_{\beta_1} e : \forall a : A_v \rightarrow^{\alpha_2(v)}_{\beta_2(v)} B \quad \Gamma \vdash^{\alpha_3}_{\beta_3} a : A}{\Gamma \vdash^{\alpha_1+\alpha_2(\beta_3)+\alpha_3}_{\beta_2(\beta_3)} e\ a : B} \; app$$

Please note that notation $\forall x_v : A \rightarrow^{\alpha(v)}_{\beta(v)} B$ has a size variable v declared as a depth of term variable x, and then bound in polynomials $\alpha(v)$ and $\beta(v)$

The notation $\alpha(1)$ is a shortcut for $\alpha[\![1/v]\!]$ in the rules *abs* and *app*.

Recursion

$$\frac{\Gamma \vdash^{\alpha_1}_{\beta_1} f : A_v \to^{\alpha_2(v_1)}_{\beta_2(v_2)} A \quad \Gamma \vdash^{\alpha_3}_{\beta_3} k : B \quad \Gamma \vdash^{\alpha_4}_{\beta_4} a : A}{\Gamma \vdash^{\alpha_1 + \alpha_3 + iter(\alpha_2, \beta_3, v_1)[\![\beta_4 / v_1]\!] + \alpha_4 + 1}_{\beta_1 [\![iter(\beta_2, \beta_3, v_2)[\![\beta_4 / v_2]\!] / v]\!]} rec(f, k, a) : B} \; rec$$

$rec(f, k, a)$ iterates function f at k times over a.

Consistency

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- Every rule beside *subsume* and *rec* is present in intuitionistic logic.
- *rec fka* can be understood as k unfoldings of *app*: $f(f(..(a)))$
- Hence consistency by embedding in IL: all *instance* of the statements can be reduced to finite IL proofs

Expressivity

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- Bounds computable in advance - at least primitive recursive
- Bounded Post-Turing machine

Emulation completeness

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Theorem

Assume a time complexity $c(x)$ for program (or proof) s that can be encoded as CUFL bounds. Iff we can emulate (encode evaluation) of $f(x)$ with an overhead e for each step, then we can prove that complexity of evaluating s is $e * c(x) + cc(x)$.

Post-Turing machine step can be easily emulated in $log^2(|\Sigma|)$.

Proof of emulation completeness

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Proof.

Assuming that $e(f)$ is function emulation in CUFL, we can write proof expression $iter(e(f), e(c), x)$. This expression evaluated encoded s and has exactly the assumed complexity

$\square$

Meta-reasoning

Meta-reasoning

We can encode bounds as terms:

$$
\begin{array}{lcl}
\mathsf{Var}_\beta & = & \mathsf{Nat}_\beta \\
\mathsf{Bound}_{\beta+1} & = & \mathsf{Var} \vee \mathsf{Nat}_\beta \vee \circ \vee (\mathsf{Bound}_\beta, \mathsf{Bound}_\beta) \\
 & \vee & (\mathsf{Bound}_\beta, \mathsf{Bound}_\beta) \\
 & \vee & (\mathsf{Bound}_\beta, \mathsf{Bound}_\beta) \vee (\mathsf{Bound}_\beta, (\mathsf{Bound}_\beta, \mathsf{Var})) \\
 & \vee & (\mathsf{Bound}_\beta, (\mathsf{Var}, \mathsf{Bound}_\beta)) \\
[\![v]\!] & = & in_l(in_l(in_l(\mathbb{B}\,(v)))) \\
[\![i]\!] & = & in_l(in_l(in_r(i))) \\
[\![()]\!] & = & in_l(in_l(in_r(()))) \\
[\![\rho_1 + \rho_2]\!] & = & in_l(in_r(in_r(([\![\rho_1]\!], [\![\rho_2]\!])))) \\
[\![\rho_1 * \rho_2]\!] & = & in_r(in_l(in_l(([\![\rho_1]\!], [\![\rho_2]\!])))) \\
[\![\rho_1^{\rho_2}]\!] & = & in_r(in_l(in_r(([\![\rho_1]\!], [\![\rho_2]\!])))) \\
[\![iter(\rho_1, \rho_2, v)]\!] & = & in_r(in_r(in_l(([\![\rho_1]\!], ([\![\rho_2]\!], \mathbb{B}\,(v)))))) \\
[\![\rho_1\ [\![\rho/v]\!]]\!] & = & in_r(in_r(in_r(([\![\rho_1]\!], ([\![\rho_2]\!], \mathbb{B}\,(v))))))
\end{array}
$$

Meta-reasoning 2

Meta-reasoning 2

We can also encode types:

$$
\begin{array}{lcl}
[\![A \vee B]\!] & = & in_l(in_l(([\![A]\!], [\![B]\!]))) \\
[\![A \wedge B]\!] & = & in_l(in_r(([\![A]\!], [\![B]\!]))) \\
[\![\forall x_v : A \to^{\alpha}_{\beta} B]\!] & = & in_r(in_l((\lambda x : A.[\![B]\!], (\lambda v : \mathsf{Nat}_v . [\![\alpha]\!], \lambda v : \mathsf{Nat}_v . [\![\beta]\!])))) \\
[\![\circ]\!] & = & in_r(in_r(()))
\end{array}
$$

Meta-reasoning 3

Meta-reasoning 3

Finally we can encode the proof terms:

$$
\begin{array}{lcl}
[\![x_v]\!] & = & in_l(in_l(in_l(in_l((\mathbb{B}\,(x), v))))) \\
[\![subsume(A, B)]\!] & = & in_l(in_l(in_l(in_r(([\![B]\!]_{\text{Bound}}, [\![A]\!]))))) \\
[\![unit]\!] & = & in_l(in_l(in_r(in_l(())))) \\
[\![in_l(A)]\!] & = & in_l(in_l(in_r(in_r(A)))) \\
[\![in_r(A)]\!] & = & in_l(in_r(in_l(in_l(A)))) \\
[\![prj_l A]\!] & = & in_l(in_r(in_l(in_r(A)))) \\
[\![prj_r A]\!] & = & in_l(in_r(in_r(in_l(A)))) \\
[\![(A, B)]\!] & = & in_l(in_r(in_r(in_r(([\![A]\!], (interpB, ,)))))) \\
[\![app(A, B)]\!] & = & in_r(in_l(in_l(in_l(([\![A]\!], [\![B]\!]))))) \\
[\![abs\ \lambda x_v.A]\!] & = & in_r(in_l(in_l(in_r(((\mathbb{B}\,(x), \mathbb{B}\,(v)), [\![A]\!]))))) \\
[\![rec]\!] & = & in_r(in_l(in_r(in_l((([\![A]\!], [\![B]\!]), \mathbb{B}\,(v))))))
\end{array}
$$

Meta-reasoning 4

Meta-reasoning 4

Theorem

Emulation completeness of ULF can be proven in itself.

The naive interpreter can be improved by applying CPS transformation, $O(\lg(|v|))$ dictionary lookups, and higher-order abstract syntax to avoid traversing entire term on substitution.

Problems stated

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2. Then we are guaranteed that we have an answer within given time.
3. We may likewise bound computation of bounds.

Avoiding infinities

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- for example: every statement about **all** natural numbers is statement about infinity
- avoid transfinite ordinals

Consequences

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- Bounded time-to-answer (*when*)
- Redefines logical expressivity
- Redefines decidability (what **and** when)
- Only computable functions
- Avoids semidecidability paradox

Future work

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- List paradoxes reverted

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- Theorem prover

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- List paradoxes reverted
- Theorem prover
- Amortized cost 1

Conclusion

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References

References I

Dummett, Michael. 1975. "Wang's Paradox." *Synthese* 30 (3/4): 301–24. http://www.jstor.org/stable/20115034.

Gisin, Nicolas. 2019. "Indeterminism in Physics, Classical Chaos and Bohmian Mechanics. Are Real Numbers Really Real?" https://arxiv.org/abs/1803.06824.

Gorelik, Gennady. 2010. "Bremermann's Limit and cGh-Physics." https://arxiv.org/abs/0910.3424.

Krajicek, Jan. 1995. *Bounded Arithmetic, Propositional Logic and Complexity Theory*. Encyclopedia of Mathematics and Its Applications. Cambridge University Press. https://doi.org/10.1017/CBO9780511529948.

Krauss, Lawrence, and Glenn Starkman. 2004. "Universal Limits on Computation," May.

Lloyd, Seth. 2002. "Computational Capacity of the Universe." *Physical Review Letters* 88 23: 237901.

Magidor, Ofra. 2007. "Strict Finitism Refuted?" *Proceedings of the Aristotelian Society* 107 (1pt3): 403–11. https://doi.org/10.1111/j.1467-9264.2007.00230.x.

References II

Meyer, Albert R., and Dennis M. Ritchie. 1967. "The Complexity of Loop Programs." In *Proceedings of the 1967 22nd National Conference*, 465–69. ACM '67. New York, NY, USA: Association for Computing Machinery. https://doi.org/10.1145/800196.806014.

Sazonov, Vladimir Yu. 1995. "On Feasible Numbers." In *Logic and Computational Complexity*, edited by Daniel Leivant, 30–51. Berlin, Heidelberg: Springer Berlin Heidelberg.

Schirn, Matthias, and Karl-Georg Niebergall. 2005. "Finitism = PRA? On a Thesis of w. W. Tait." *Reports on Mathematical Logic*, January.

Troelstra, A. S. 1988. *Constructivism in Mathematics: An Introduction*. Elsevier. https://www.sciencedirect.com/bookseries/studies-in-logic-and-the-foundations-of-mathematics/vol/121/suppl/C.

Zeilberger, Doron. 2010. "Opinion 108: …the Feeling Is Mutual: I Feel Sorry for Infinitarian Hugh Woodin for Feeling Sorry for Finitists Like Myself! (And the "Lowly" Finite Is MUCH More Beautiful Than Any "Infinite")." 2010. https://sites.math.rutgers.edu/~zeilberg/Opinion108.html.