Bijections between planar maps and planar linear normal λ-terms with connectivity condition

Wenjie Fang
LIGM, Université Gustave Eiffel
arXiv:2202.03542

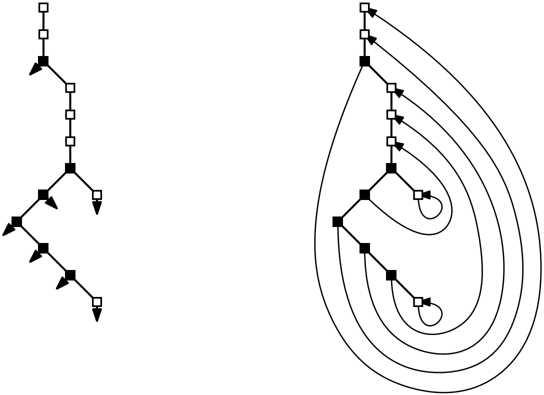
12 January 2023, CLA 2023, LIX, École polytechnique

λ-terms, according to a combinatorialist

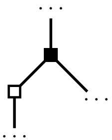
λ-term: unary-binary tree (skeleton) + variable-abstraction map

Linear λ-term: the variable-abstraction map being bijective (so closed)

$$t = \lambda u.\lambda v.u(\lambda w.\lambda x.\lambda z.v(w(x(\lambda y.y)))z(\lambda k.k))$$



Some special properties

- closed: the variable-abstraction map is complete
- unitless: no closed sub-term
- normal: no β-reduction, *i.e.*, avoiding 
- (RL-)planar: right-to-left variable-abstraction map
  Example: $t = \lambda u.\lambda v.u(\lambda w.\lambda x.\lambda z.v(w(x(\lambda y.y)))z(\lambda k.k))$

All can be translated combinatorially to trees!

Linear + planar : unique choice, so just unary-binary tree!

Known enumerations

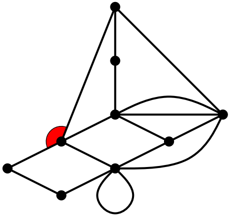
λ-terms	Maps	OEIS
linear	general cubic	A062980
planar	planar cubic	A002005
unitless	bridgeless cubic	A267827
unitless planar	bridgeless planar cubic	A000309
β-normal linear/$\sim$	general	A000698
β-normal planar	planar	A000168
β-normal unitless linear/$\sim$	bridgeless	A000699
β-normal unitless planar	bridgeless planar	A000260

Noam Zeilberger, *A theory of linear typings as flows on 3-valent graphs*, LICS 2018

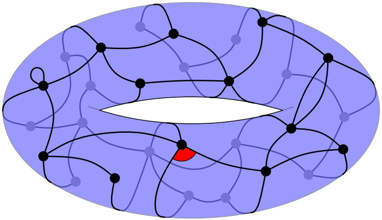
A lot of people and work: Zeilberger, Bodini, Gardy, Jacquot, Giorgetti, Courtiel, Yeats, . . .

What is a map?

Combinatorial map: drawing of graphs on a surface



sphere/planar ($g = 0$)



torus ($g = 1$)

We only consider rooted map, *i.e.*, with a marked corner.

Notions in graph theory: planar, cubic, bridgeless, loopless, bipartite, ...

Connectivity condition

Noam Zeilberger and Jason Reed (CLA 2019)

How about connectivity of the diagram on **planar linear normal terms**?

k-connected: breaking $k-1$ edges does not split the graph

- 1-connected: all (connected by their skeleton)
- 2-connected: unitless (bridge $\Leftrightarrow$ closed sub-term)
- 3-connected: ???

Conjecture (Zeilberger–Reed, 2019)

The number of 3-connected planar linear normal λ-terms with $n+2$ variables is

$$\frac{2^n}{(n+1)(n+2)}\binom{2n+1}{n},$$

which also counts bipartite planar maps with n edges (A000257).

Our contribution (1)

Katarzyna Grygiel and Guan-Ru Yu (CLA 2020): combinatorial characterization of 3-connected terms, partial bijective results

Theorem (Fang, 2023+)

There is a direct bijection between 3-connected planar linear normal λ-terms with $n+2$ variables and bipartite planar maps with n edges.

What we do using bijections:

- Transfer of statistics (also about applications in λ-terms)
- Generating functions and probabilistic results also for free!

Proposition (F. 2023+, from known results on maps)

Let $X_n = \#$ initial abstractions of a uniformly random 3-connected planar linear normal λ-term. When $n \to \infty$,

$$\mathbb{P}[X_n = k] \to \frac{k-1}{3}\binom{2k-2}{k-1}\left(\frac{3}{16}\right)^{k-1}.$$

Our contribution (2)

Theorem (Fang, 2023+)

There is a direct bijection from planar linear normal λ-terms to planar maps. Furthermore, when restricted to unitless terms, the bijection leads to loopless planar maps.

λ-terms	Maps	OEIS
β-normal linear/$\sim$	general	A000698
β-normal planar	planar	A000168
β-normal unitless linear/$\sim$	bridgeless	A000699
β-normal unitless planar	loopless planar	A000260

Known recursive bijection in (Zeilberger and Giorgetti, 2015) via LR-planar terms

From λ-terms to unary-binary trees

Linear planar λ-terms $\Leftrightarrow$ unary-binary trees (with conditions)

Three statistics for a unary-binary tree S:

- $\mathrm{unary}(S)$: number of unary nodes (abstractions)
- $\mathrm{leaf}(S)$: number of leaves (variables)
- $\mathrm{excess}(S)$: $\mathrm{leaf}(S) - \mathrm{unary}(S)$ (free variables)

Properties of linear planar λ-terms $\Leftrightarrow$ properties on skeletons

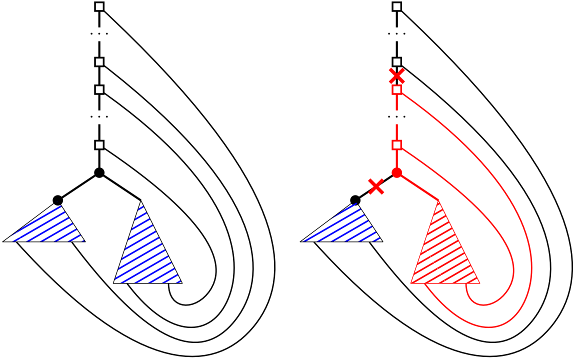
S_u: sub-tree of S induced by u

- Linear (or closed) $\Leftrightarrow$ $\mathrm{excess}(S) = 0$
- Normal $\Leftrightarrow$ Left child of a binary node is never unary
- 1-connected $\Leftrightarrow$ $\mathrm{excess}(S_u) \geq 0$ for all u
- 2-connected $\Leftrightarrow$ $\mathrm{excess}(S_u) > 0$ for all u non-root

Characterization of 3-connectedness (1)

Proposition (Grygiel and Yu, CLA 2020)

Let S be the skeleton of a 3-connected planar linear λ-term, then the left child of the first binary node is a leaf.



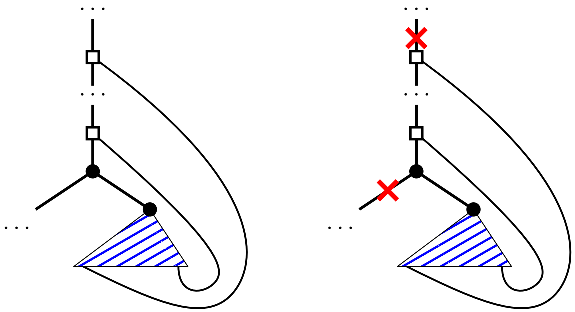
Reduced skeleton: the right sub-tree of the first binary node

Characterization of 3-connectedness (2)

Proposition (Proposed by Grygiel and Yu, CLA 2020)

S is the reduced skeleton of a 3-connected planar linear normal λ-term iff

- **(Normality)** *The left child of a binary node in S is never unary;*
- **(3-connectedness)** *For every binary node u with v its right child, $\mathrm{excess}(S_v)$ is strictly larger than the number of consecutive unary nodes above u.*



Clearly necessary, but also sufficient!

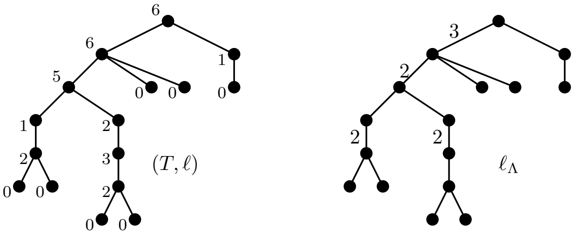
Degree trees

Degree tree: a plane tree T with a labeling ℓ on nodes with

- u is a leaf $\Rightarrow \ell(u) = 0$;
- u has children $v_1, \ldots v_k \Rightarrow s(u) - \ell(v_1) \leq \ell(u) \leq s(u)$, where $s(u) = k + \sum_{i=1}^{k} \ell(v_i)$.

Contribution of each child : 1 (itself) + $\ell(v_i)$ (its sub-tree)

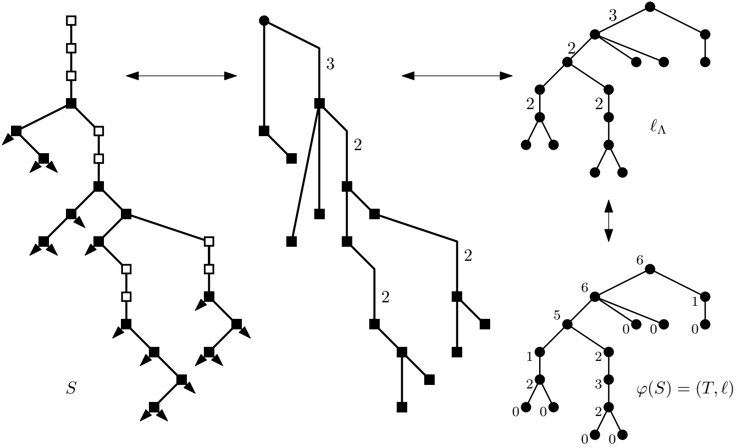
Except for the first child: from 1 to its due contribution.



Edge labeling ℓ_Λ: the subtracted contribution

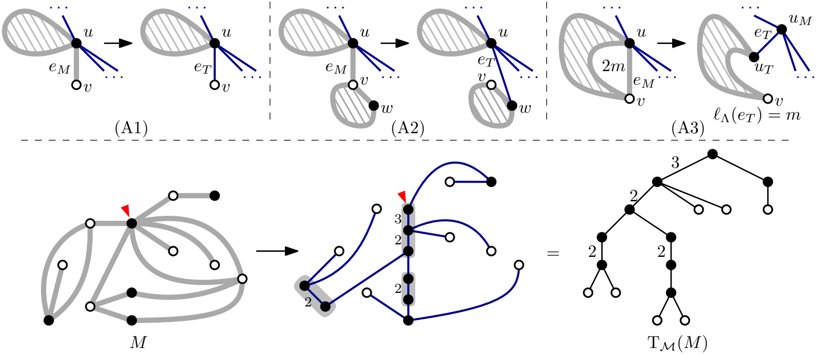
ℓ and ℓ_Λ interchangeable!

First bijection (1/2): 3-connected terms ⇔ degree trees



- Unary nodes on right child ⇔ Subtraction on left child
- Leftmost leaf ⇔ Contribution 1

First bijection (2/2): degree trees ⇔ bipartite planar maps



Existing direct bijection (F., 2021), using an exploration

Also related to Chapoton's new intervals in the Tamari lattice

Some statistics correspondences:

- Unary chains of length k ⇔ edge label k ⇔ faces of degree $2k$
- Initial unary chain ⇔ root label ⇔ degree of root face

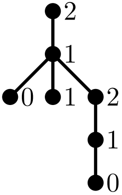
Connected terms and trees

Recall the conditions:

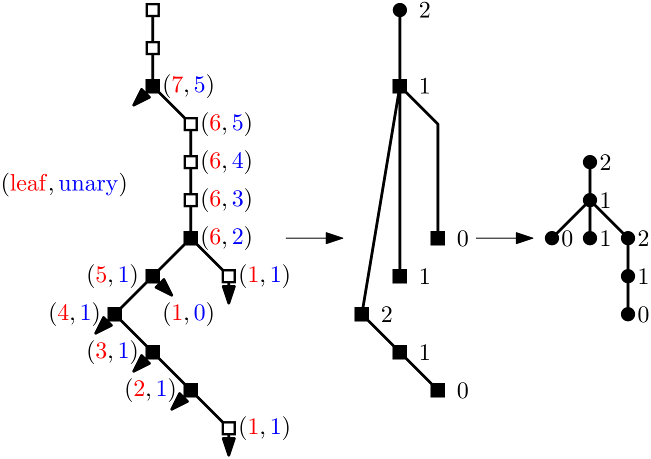
- 1-connected $\Leftrightarrow \mathrm{excess}(S_u) \geq 0$ for all u
- 2-connected $\Leftrightarrow \mathrm{excess}(S_u) > 0$ for all u non-root

v-trees: a plane tree T with a labeling ℓ on nodes with

- Leaves $u \Rightarrow \ell(u) \in \{0, 1\}$;
- Non-root u with children $v_1, \ldots, v_k \Rightarrow 0 \leq \ell(u) \leq 1 + \sum_{i=1}^{k} \ell(v_i)$



Second bijection (1/2)



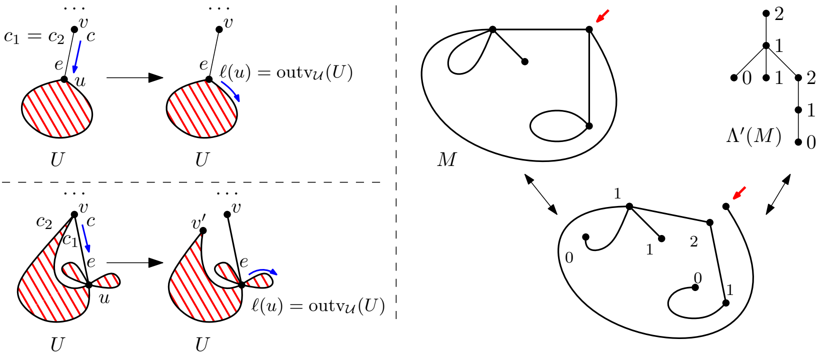
$\lambda u.\lambda v.u(\lambda w.\lambda x.\lambda z.v(w(x(\lambda y.y)))z(\lambda k.k))$

Excess of the **right child**! $0 \Leftrightarrow$ closed sub-term

Second bijection (2/2)

Direct bijection = “de-recusifyng” a new recursive decomposition

$\mathrm{outv}_{\mathcal{U}}$: #vertices on outer face -1



Loop ⇔ component with 1 outer node ⇔ label 0 in tree

Natural specialization to loopless planar maps ⇔ unitless terms!

Recapitulation

λ-terms	Maps	OEIS
β-normal linear/$\sim$	general	A000698
β-normal planar	planar	A000168
β-normal unitless linear/$\sim$	bridgeless	A000699
β-normal unitless planar	loopless planar	A000260
β-normal 3-connected planar	bipartite planar	A000257

First bijection (direct) for 3-connected planar normal terms and bipartite planar maps

New bijection (direct) for general planar normal terms and planar maps, naturally restricted to 2-connected terms and loopless planar maps

Not the same bijection... But in the same spirit.

Higher connectivity? Other enumeration consequences?

Recapitulation

λ-terms	Maps	OEIS
β-normal linear/$\sim$	general	A000698
β-normal planar	planar	A000168
β-normal unitless linear/$\sim$	bridgeless	A000699
β-normal unitless planar	loopless planar	A000260
β-normal 3-connected planar	bipartite planar	A000257

First bijection (direct) for 3-connected planar normal terms and bipartite planar maps

New bijection (direct) for general planar normal terms and planar maps, naturally restricted to 2-connected terms and loopless planar maps

Not the same bijection... But in the same spirit.

Higher connectivity? Other enumeration consequences?

Thank you for listening!