Bijections between planar maps and planar linear normal λ-terms with connectivity condition

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November 16, 2022

As a well-known Turing-complete computational model, λ-calculus has been well-studied in logic and related fields. However, the enumeration of λ-terms did not attract attention in combinatorics until relatively recently. One of the most well-studied family is that of *linear λ-terms*, due to its connections with combinatorial maps, which are graph embeddings on surfaces. Such connections were pioneered by Bodini, Gardy and Jacquot in [BGJ13], where a simple bijection between linear λ-terms and cubic maps was given. Due to its simplicity, this bijection transfers interesting statistics and can be specialized to interesting sub-families. This approach has also led to the study of asymptotic properties and statistics distribution of related λ-terms (see, *e.g.*, [BGJ13, BGGJ13, BSZ21]).

Independently, Zeilberger and Giorgetti found in [ZG15] a bijection between a sub-family of linear λ-terms called *planar linear normal λ-terms* and planar maps. When restricted to λ-terms that are also *unitless*, *i.e.*, without closed sub-terms, this bijection leads to bridgeless planar maps. The unitless condition here is equivalent to the 2-connectedness of the syntactic diagram of the λ-term. Such connections lead naturally to the consideration of higher connectivity conditions on planar linear λ-terms.

In a talk at CLA 2019 [ZR19], Zeilberger and Reed considered λ-terms with higher connectivity. Based on computations, they proposed the following conjecture.

Conjecture 1 ([ZR19])**.** *The number of 3-connected planar linear normal λ-terms with $n+2$ variables is equal to that of bipartite planar maps with n edges [Tut63], which is*

$$\frac{2^n}{(n+1)(n+2)}\binom{2n+1}{n}.$$

In a talk at CLA 2020 [GY20], Grygiel and Yu tried to relate these 3-connected λ-terms to $\beta(0,1)$-trees, which are in bijection with bicubic planar maps [CS03] and thus with bipartite planar maps [Tut63], and succeeded in a few special cases. The key of their work is a characterization of skeletons of such λ-terms.

In this article, based on the characterization given by Grygiel and Yu in [GY20], we prove Conjecture 1 by giving a bijection between 3-connected planar linear normal λ-terms and bipartite planar maps, passing through the so-called *degree trees* defined in [Fan21]. We have the following theorem.

Theorem 2. *For $n \geq 2$, there is a bijection between 3-connected planar linear normal λ-terms with n variables and bipartite planar maps with $n-2$ edges.*

Using this new bijection, we study the refined enumeration of such λ-terms under several statistics, and also the asymptotic distribution of some statistics. As an example, in [Lis99], Liskovets studied the asymptotic vertex degree distribution of many families of planar maps, including Eulerian planar maps, which are duals of bipartite planar maps. Using our bijection, we translate results in [Lis99] to the following corollary.

Corollary 3. *Let X_n be the number of abstractions at the beginning of a λ-term chosen uniformly from $\mathcal{C}_n^{(3)}$. Then, for $k \geq 2$, when $n \to \infty$, we have*

$$\mathbb{P}[X_n = k] \to \frac{k-1}{3}\binom{2k-2}{k-1}\left(\frac{3}{16}\right)^{k-1}.$$

With a similar approach, we also give a new bijection between planar linear normal λ-terms and planar maps, which is direct and also has a nice restriction to 2-connected λ-terms. As a direct bijection, it may be a better tool to explore the relation between planar maps and related λ-terms.

Theorem 4. *There is a direct bijection between planar linear normal λ-terms with n variables and planar maps with $n-1$ edges. Furthermore, it sends those λ-terms that are also 2-connected to loopless planar maps.*

For this bijection, we define a family of node-labeled trees called *v-trees*, which can be seen as description trees (see [CS03]) of planar maps under a seemingly new recursive decomposition called *one-corner decomposition*. We then give direct bijections from v-trees to both planar maps and planar linear normal λ-terms, thus linking the two families. Our bijection is different from that in [ZG15], even after taking the map dual, which sends loopless planar maps to bridgeless planar maps.

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