Son of the Vagabond

Lambda Calculus

Application   X,Y ~> (X,Y)
        and its inverse
Abstraction   x,X ~> \x.X
        where
Beta   (\x.X Y) red. [Y/x]X
Eta   \x.(Xx)   red. X
        give the congruence beta-eta
[Church 1941]

Combinators

Bxyz red. x(yz)

Cxyz red. xzy

Ix red. x

Kxy red. x

Wxy red. xyy

[Curry 1958]

Of course for equivalence we need the strong reduction .

[Hindley 1967]

Church's Monoid

product = B = \xyz. x(yz) = *

identity = I = \x.x

add axioms for enough beta-eta to make a monoid plus

B(Bxy) conv. B(Bx)(By)

B(x*y) = Bx * By

BB(Bx) conv. B(B(Bx))B

B * Bx = B(Bx) * B

[Statman 1988]

Richard Thompson's Group F

Although finitely generated an infinite presentation on the generators

$$\{g(0),g(1),...,g(n),...\}$$

is

$$g(k)^{-1} * g(n) * g(k) = g(n+1)$$

for all n and k < n.

[Thompson 1962]

The B,I Monoid Generates F

From Curry & Feys we set

$$B^{0} = B \qquad B^{n+1} = B(B^{n})$$

$$B_{1} = B \qquad B_{n+1} = B*B_{n}$$

We say that M is well written if M =

$$B^{n(1)}*B^{n(2)}*...*B^{n(k)}$$

where $n(i)+1 > n(i+1)$ for i =1,...,k or

M = I.

Cancellation

Now it is easy to prove that for each M there exists a unique well written N s.t. M conv.N. From this it follows that the B,I monoid is a cancellation monoid. It generates the group F where $g(n) = B^{n}$.

Surjective Pairing

Church's delta function was only for beta normal forms, and Klop showed that surjective pairing

L([x,y]) red. x

R([x,y]) red. y

[Lx,Rx] red. x

is not Church-Rosser. Nevertheless, de Vrijer showed it is consistent.

[Stovring 2006]

Cartesian Monoids

Now lift to functions pointwise

<x,y> = \z. [xz,yz]

L*<x,y> = x

R*<x,y> = y

<L,R> = I

<x,y>*z = <x*z,y*z>

using eta conversion.

This is also a fragment of Backus' FP

[Statman 1996, 2018]

The Free Cartesian Monoid

Let CM be the free Cartesian monoid.
W is well written if W is a member of the sub-monoid generated by L and R or W = <W(1),W(2)> where W(1) and W(2) are well written. Well written expressions can be contracted and expanded by

<L*x,R*x> cont. x

<L,R> cont. I

The Group of CM

A well written W can be thought of as a binary tree with strings of L's and R's at its leaves.

For each element M of CM there exists a well written W s.t. M = W. Moreover, W is unique modulo expansions and contractions.

It is easy to prove that a well written W is a member of the group of CM iff W can be expanded and contracted to a well written

Intermezzo

W ' s.t. the tree of W ' has exactly 2^{n} leaves and every string of L's and R's of length n occurs (exactly once). Note no restriction on the shape of the tree. This is because F is hidden in the condition. As a simple consequence, the group has elements of infinite order.

Richard Thompson's Group V

The Vagabond group is most intuitively defined as the subgroup of the homeomorphism group of Cantor space consisting of those that are “step” functions (piecewise shift operators). A map F from Cantor Space into Cantor Space is said to be a piecewise shift operator if for all [Statman 1996]

Piecewise Shift Operators

f there exists g and finite u,v s.t .

(1) f = u^g

(2) F(f) = v^g

(3) for any h, F(u^h) = v^h.

Here f,g,h are infinite binary sequences.

Now we set

I(f) = f

L(f) = 0^f

R(f) = 1^f

V is the Group of CM

<F,G>(0^f) = F(f)

<F,G>(1^f) = G(f)

(F*G)(f)   = G(F(f))

And the set of piecewise shift operators form a Cartesian monoid. It follows easily that the group of CM is V.

Mariangeola Dezani's Group

An intuitive description. Start with a group G. Consider sequences g,h of elements from G with finite support. Permutations p,q of the natural numbers with finite support. Product

(p,g)*(q,h) = (p*q, \i. h(i)*g(p(i))).

Now begin with the trivial group and iterate. Finally take a direct limit (Barendregt pg. 546).

Dezani's Theorem

The lambda terms with two sided inverses are the hereditary permutations.

Theorem: (Dezani 1976)

The group of the lambda calculus monoid under beta-eta conversion is Dezani's group.

Theorem :(Klop 1980?)

Finitely generated subgroups of Dezani's group are finite.

What is this group ?

V cannot even be embedded into Dezani's group.

Open Problem:

What is the group of

lambda beta-eta + surjective pairing ?

Collatio Tomi

An application to group theory – sort of.

Recently we were able to use the Cartesian monoid representation of V to show that the membership problem for finitely generated subgroups of V is decidable.

(ACT 2020)