Limiting probabilities of first order properties in sparse random graphs

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First order logic (FO) of graphs

Quantifiers: $\forall, \exists$
Variables: $x, y, z, \ldots$
Boolean connectives and equality: $\vee, \wedge, \neg, \rightarrow, =$
Predicates $P(x)$, $Q(x, y)$, ...

$E(x, y)$ adjacency relation written $x \sim y$
Assumed symmetric and antireflexive

Some examples

- Existence of a triangle: $\exists x \, \exists y \, \exists z \; (x \sim y) \wedge (y \sim z) \wedge (z \sim x)$
- Existence of fixed H as a subgraph
- There are at most a cycles of length at most k

FO cannot express connectivity, planarity, 3-colorability...

The $G(n, p)$ model of random graphs

$G(n, p)$ with $0 < p < 1$

Vertices: $V = \{1, 2, \ldots, n\}$
Each edge $\{i, j\}$ is in $G(n, p)$ independently with probability p

$$\mathbf{P}(G) = p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|}$$

The expected number of edges is $p\binom{n}{2} \sim p\frac{n^2}{2}$

p constant — Dense graphs: $\Theta(n^2)$ edges

$p = \dfrac{c}{n}$ — Sparse graphs: $\Theta(n)$ edges

Sparse random graphs

$G_n = G(n, c/n)$

The phase transition Erdős-Rényi (1960)

- For $c < 1$, all components in G_n are either trees or have a unique cycle, and have size $O(\log n)$
- For $c > 1$ there is a unique component of size $\Theta(n)$

FO logic cannot capture the transition since it cannot express that a graph is acyclic or unicyclic

Limiting probabilities

$G_n = G(n, c/n)$

Given a FO property A we are interested in the limiting probability

$$\lim_{n\to\infty} \mathbf{P}[G_n \text{ satisfies } A]$$

Lynch 1992
For every FO property A

$$\lim_{n\to\infty} \mathbf{P}[G_n \text{ satisfies } A\,] = f_A(c)$$

and $f_A(c)$ is an expression in c using only $+$, $\times$, $\{\lambda\colon \lambda \in \Lambda(c)\}$ and exponentials, hence it is a C^∞ function

Remark Alberto Larrauri has recently generalized Lynch's result to sparse hypergraphs [Journal of Logic and Computation, to appear]

The set of limiting probabilities for sparse graphs

$L_c = \{\lim \mathbf{P}\left[G\left(n, \frac{c}{n}\right) \text{ satisfies } A\right] : \text{FO property } A\}$

L_c is a countable set, we consider its topological closure

$$\overline{L_c} \subseteq [0,1]$$

Theorem [Alberto Larrauri, Tobias Müller, M.N.]

Let $c_0 \approx 0.9368$ be the unique positive root of

$$e^{\frac{c}{2}+\frac{c^2}{4}}\sqrt{1-c} = \frac{1}{2}$$

- $\overline{L_c}$ is a finite union of intervals
- For $c \geq c_0$ we have $\overline{L_c} = [0,1]$
- For $0 < c < c_0$ there is at least one gap in $\overline{L_c}$, that is, an interval $[a,b] \subseteq [0,1]$ with $[a,b] \cap \overline{L_c} = \emptyset$

Previous work

$\mathcal{P}$ labelled planar graphs

Heinig, Müller, N., Taraz 2018

- With the uniform distribution on graphs in $\mathcal{P}$ with n vertices $\overline{L_c}$ is a finite union of intervals (in fact 108 intervals of length $\approx 10^{-6}$)
- For the class of random forests (acyclic graphs) $\overline{L_c} = [0, 0.170] \cup [0.223, 0.393] \cup [0.606, 0.776] \cup [0.830, 1]$
- For every minor-closed class of graphs which is addable (the forbidden minors are 2-connected) $\overline{L_c}$ is finite union of intervals and there is always a gap

Sketch of proof

1. No gap for $c \geq 1$
2. At least one gap for $c < c_0$
3. No gap for $c_0 \leq c < 1$

1. No gap for $c \geq 1$

X_k = number of k-cycles in $G(n, \frac{c}{n})$

$$X_k \xrightarrow[n\to\infty]{\text{d}} \text{Poisson}\left(\frac{c^k}{2k}\right)$$

and $X_3, \ldots, X_k$ are asymtotically independent for fixed k

$$X_{\leq k} := X_3 + \cdots + X_k \xrightarrow[n\to\infty]{\text{d}} \text{Poisson}\left(\mu_k = \sum_{i=3}^{k} \frac{c^k}{2k}\right)$$

- $\lim_{k\to\infty} \mu_k = \infty$ since $c \geq 1$
- The property $\{X_{\leq k} \leq a\}$ is FO expressible for fixed k

$$\mathbf{P}(\text{Poisson}(\mu) \leq \mu + x\sqrt{\mu}) \xrightarrow[\mu\to\infty]{} \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

Given $p \in (0,1)$ and $\epsilon > 0$ take x with $\Phi(x) = p$ and μ_0 such that

$$|\mathbf{P}(\text{Poisson}(\mu) \leq \mu + x\sqrt{\mu}) - p| < \epsilon, \qquad \text{for } \mu \geq \mu_0$$

Finally take k such that $\mu_k \geq \mu_0$

Hence $\overline{L_c} = [0,1]$

2. At least one one gap for $c < c_0$

Structure of $G_n = G(n, c/n)$ for $c < 1$
Erdős-Rènyi (1960)

- $G_n = F_n \cup H_n$

  F_n is a **forest**, H_n is a collection of **unicyclic** graphs

- For every fixed tree T and $m \geq 1$,
  F_n contains at least m copies of T

Zero-one law in F_n For every FO property A

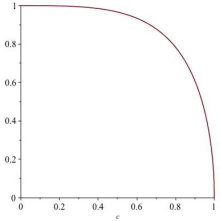
$$\lim_{n\to\infty} \mathbf{P}[F_n \text{ satisfies } A] \in \{0, 1\}$$

Idea F_n contains arbitrarily many copies of each tree, hence two random instances of F_n cannot be distinguished by FO properties

This can be proved for instance using combinatorial games (Ehrenfeucht-Fraïssé)

Let F be the property of G_n being acyclic (a forest)

$$\lim_{n\to\infty} \mathbf{P}(F) = \prod_{k\geq 3} e^{-c^k/(2k)} = e^{-\sum_{k\geq 3} \frac{c^k}{2k}} = \sqrt{1-c}\,e^{c/2+c^2/4} = f(c)$$



We have $\lim_{n\to\infty} \mathbf{P}(F) > 1/2$ for $c < c_0$
Given a FO property A, we have in terms of limiting probabilities

$$\mathbf{P}(A) = \mathbf{P}(A|F)f(c) + \mathbf{P}(A|\neg F)(1-f(c))$$

If $\mathbf{P}(A|F) = 1$ then $\mathbf{P}(A) \geq f(c) > 1/2$

If $\mathbf{P}(A|F) = 0$ then $\mathbf{P}(A) \leq 1 - f(c) < 1/2$

Hence $[1-f(c), f(c)]$ is a gap

3. No gap for $c_0 \leq c \leq 1$

- $G_n = F_n \cup H_n$
  F_n is a **forest**, H_n is a collection of **unicyclic** graphs
- $\mathbf{E}(|H_n|)$ is bounded

Restricting to F_n we have the FO zero-one law

Hence whether G_n satisfies a FO property depends solely on the fragment H_n

$$\mathbf{P}[H_n \cong H] \to p_H = f(c)\frac{(ce^{-c})^{|V(H)|}}{\mathrm{aut}(H)}$$

$$\mathbf{P}[G_n \text{ satisfies } A] \to \sum_{H \in \mathcal{H}_A} p_H$$

It follows that $\overline{L_c}$ is the collection of subsums of the series

$$\sum_{H \text{ fragment}} p_H = 1$$

Nymann-Sáenz 2000 (Kakeya 1915)

Let $\sum_{n\geq 0} p_n < +\infty$ with $p_n \geq 0$

If $p_i \leq p_{i+1} + p_{i+2} + \cdots$ (term $\leq$ tail) for all $i \geq 0$ then

$$\left\{ \sum_{i \in A} p_i : A \subset \mathbb{N} \right\} = \left[0, \sum_{n=0}^{\infty} p_n \right]$$

Order the limiting probabilities of the fragments H

$$p_0 \geq p_1 \geq p_2 \geq \cdots$$

Considering

$$\sum_{|V(H)|=k} p_H = f(c)(ce^{-c})^k \sum_{|V(H)|=k} \frac{1}{\mathrm{aut(H)}}$$

we show that $p_i \leq \sum_{j>i} p_j$ for fragments of size k ≥ 4

We complete the argument for size 3 (a triangle)

$p_0 =$ probability of being acyclic (empty fragment)

$p_0 \leq 1 - p_0$ means $p_0 \geq 1/2$ which holds because $c \geq c_0$

Sparse hypergraphs

$G^d(n,p)$ random d-hypergraph in which each d-edge has independent probability p

Take $p = \dfrac{c}{n^{d-1}}$

Sparse sin the expected number of edges is $p\binom{n}{d} = \Theta(n)$

Theorem

Let c_0 be the unique positive root of

$$\exp\left(\frac{c}{2(d-2)!}\right)\sqrt{1-\frac{c}{2(d-2)!}} = \frac{1}{2}. \qquad (1)$$

- $\overline{L_c}$ is a finite union of intervals
- $\overline{L_c} = [0,1]$ for $c \geq c_0$
- At least one gap for $0 < c < c_0$

Graphs with given degree sequence

(ongoing project with Alberto Larrauri and Guillem Perarnau)

Consider random graphs (uniform distribution) with degree sequence $\mathcal{D} = d_0(n), d_1(n), \ldots$

- $d_i(n)$ = number of vertices of degree i
- $\sum_i d_i(n) = n$
- $\lim_{n\to\infty} d_i(n)/n = \lambda_i$
- $\mu_1 = \lim_{n\to\infty} \sum_i i\lambda_i$ finite
- $\mu_2 = \lim_{n\to\infty} \sum_i i^2\lambda_i$ finite

Molloy-Reed 1995
There exists a component of size $\Theta(n)$ iff $\mu_2 - 2\mu_1 > 0$

$\overline{L_{\mathcal{D}}}$ the closure of the set of limiting probabilities of FO properties for random graphs with degree sequence $\mathcal{D}$

When is $\overline{L_{\mathcal{D}}} = [0, 1]$?

Partial results so far indicate a behavior generalizing what we have found for $G(n, p)$