In search of a bijection between β-normal 3-indecomposable planar lambda terms and $\beta(0, 1)$-trees

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Computational Logic and Applications 2020

Last year in Versailles (CLA 2019)

Some topological properties of planar lambda terms
by Noam Zeilberger and Jason Reed

results & questions

3-indecomposable planar terms are counted by A000260, which also counts β-normal 2-indecomposable (= unitless) planar terms. Indeed, 3-indecomposable planar terms admit a direct inductive characterization...

$$t ::= x \mid C\{t\}$$
$$C ::= \lambda x.C \mid \bullet\, u$$

isomorphic to a similar characterization of β-normal unitless planar terms.

Conjecture: β-normal 3-indecomposable planar terms are counted by A000257!

What about non-planar 3-indecomposable terms?

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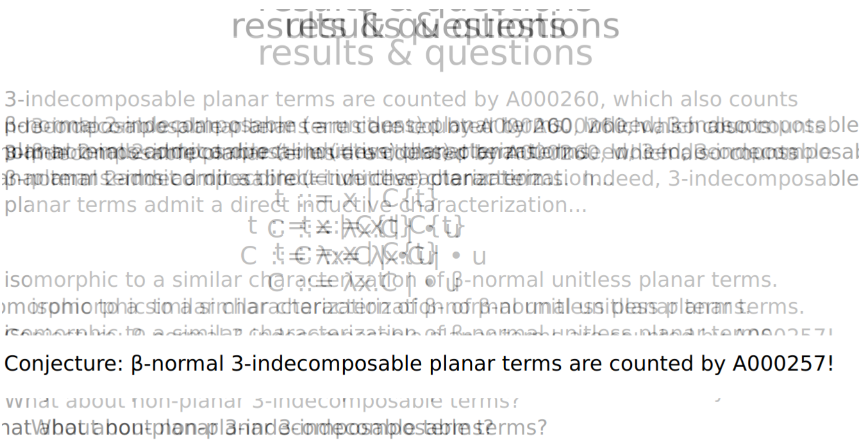

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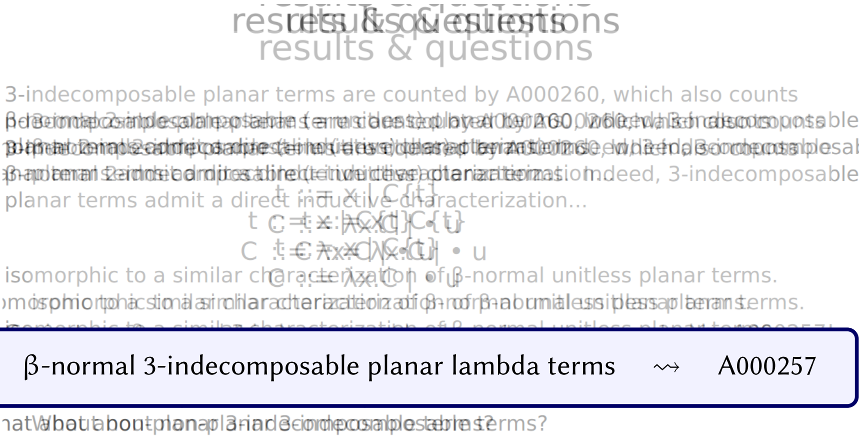

β-normal 3-indecomposable planar lambda terms ⇝ A000257

A000257

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A000257 Number of rooted bicubic maps: a(n) = (8n-4)*a(n-1)/(n+2). 21
(Formerly M2927 N1175)

1, 1, 3, 12, 56, 288, 1584, 9152, 54912, 339456, 2149888, 13891584, 91287552, 608583680, 4107939840, 28030648320, 193100021760, 1341536993280, 9390758952960, 66182491668480, 469294031831040, 3346270487838720, 23981605162844160, 172667557172477952 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,3

COMMENTS
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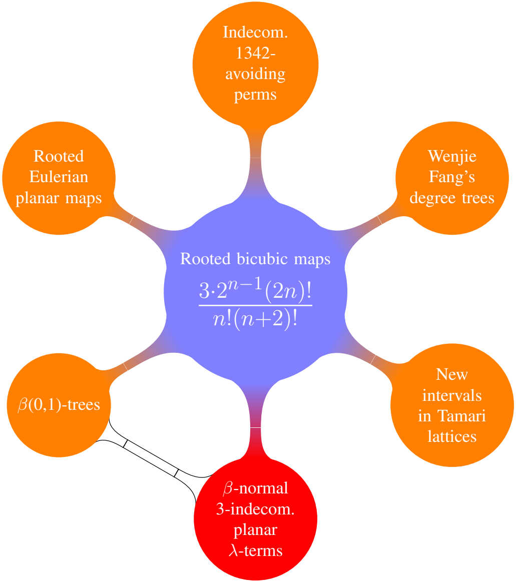
OEIS A000257



Lambda terms

Let $\mathcal{V}$ be a countable set of variables. **Lambda terms** are defined by the following grammar:

$$\mathcal{T} ::= \mathcal{V} \mid \lambda \mathcal{V}.\mathcal{T} \mid \mathcal{T}\ \mathcal{T}$$

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Example.

$$\lambda a.\,\big(\lambda b.a\big(\lambda c.(c(\lambda d.b)b)\big)\big)\;\big(a\big(\lambda e.\big(\lambda f.\lambda g.e\big)\big(\lambda h.\big((\lambda i.eh)e\big)\big(\lambda j.\lambda k.kj\big)\big)\big)\big)$$

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$$\textcolor{blue}{\lambda}.\ \big(\textcolor{green}{\lambda}.\textcolor{blue}{\bullet}\big(\textcolor{red}{\lambda}.(\textcolor{red}{\bullet}(\textcolor{brown}{\lambda}.\textcolor{green}{\bullet})\textcolor{green}{\bullet})\big)\big)\ \big(\textcolor{blue}{\bullet}\big(\textcolor{green}{\lambda}.\big(\textcolor{red}{\lambda}.\textcolor{brown}{\lambda}.\textcolor{green}{\bullet}\big)\big(\textcolor{red}{\lambda}.\big((\textcolor{brown}{\lambda}.\textcolor{green}{\bullet}\textcolor{red}{\bullet})\textcolor{green}{\bullet}\big)\big(\textcolor{red}{\lambda}.\textcolor{brown}{\lambda}.\textcolor{brown}{\bullet}\textcolor{red}{\bullet}\big)\big)\big)\big)$$

We will be interested in closed terms only.

Lambda terms as trees

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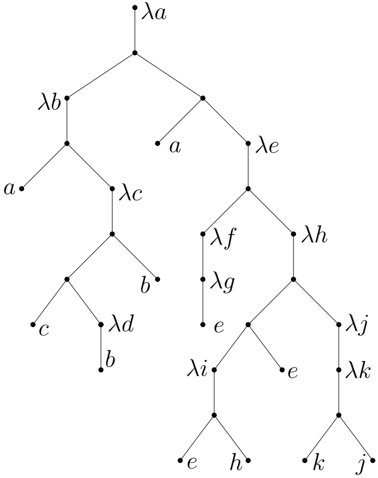

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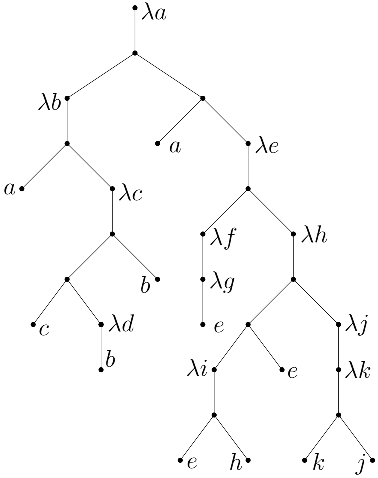

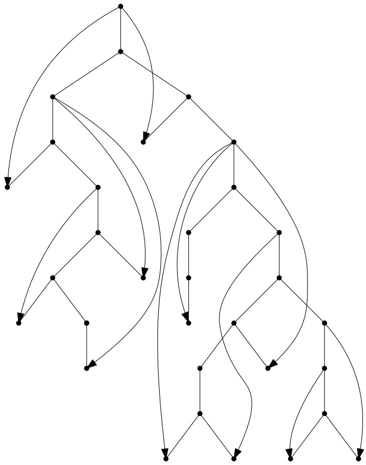

Planar lambda terms

A lambda term is **linear** iff each variable has exactly one occurrence (*i.e.*, there is a one-to-one correspondence between unary nodes and leaves). The **size** of a linear term is defines as the number of applications (binary nodes).

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Let us consider unary-binary trees with the same number of leaves and unary nodes.

Now, in the right-to-left order on leaves, we add a pointer to each leaf linking it with the closest unary node above that has not been chosen yet. The resulting structure has a planar representation on the plane.

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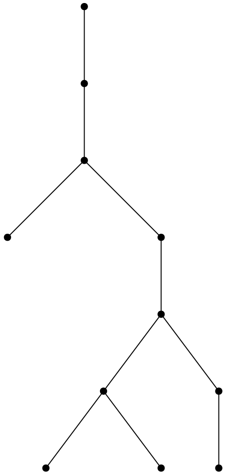
Due to planarity, we no longer need to bother about the pointers, as every unary-binary tree corresponds to at most one planar term.

Example of a planar lambda term



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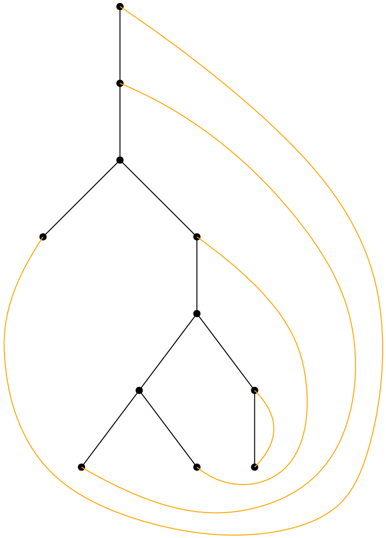

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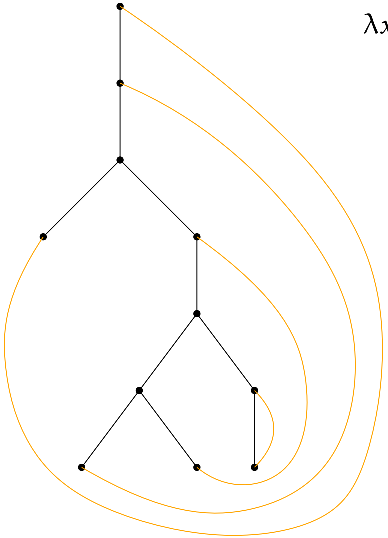

$$\lambda xy.x\big(\lambda z.(zy)(\lambda w.w)\big)$$

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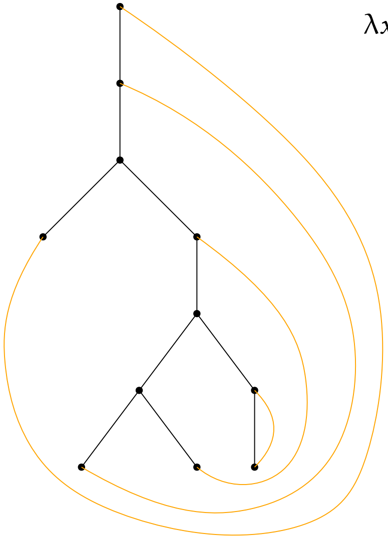

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size $= 3$

β-normal 3-indecomposable planar lambda terms

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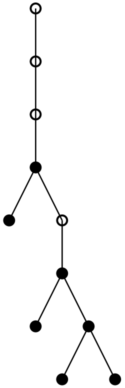
In the obtained structure, let us remove every vertex that corresponds to some leaf by contracting its two incident edges.

If the resulting map is internally 3-connected, *i.e.*, removing any two edges but the two incident to the root does not make the map disconnected, then we say that the original planar term is **3-indecomposable**.

β-normal 3-indecomposable planar lambda terms



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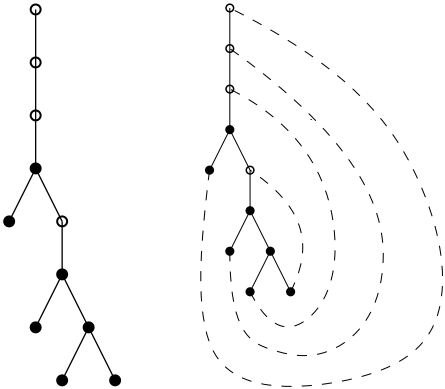

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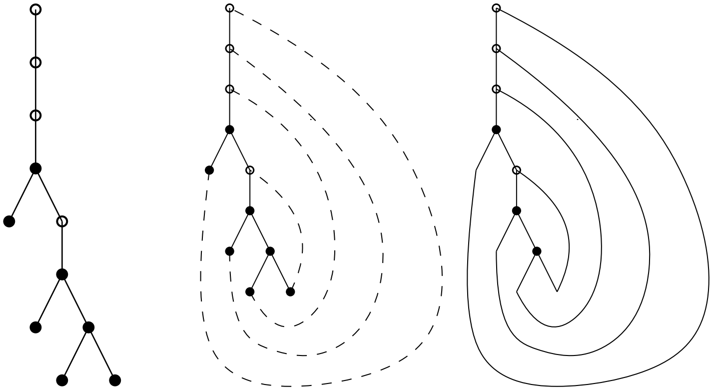
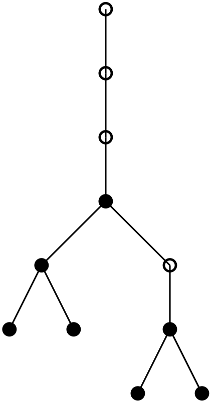


Term violating 3-indecomposability



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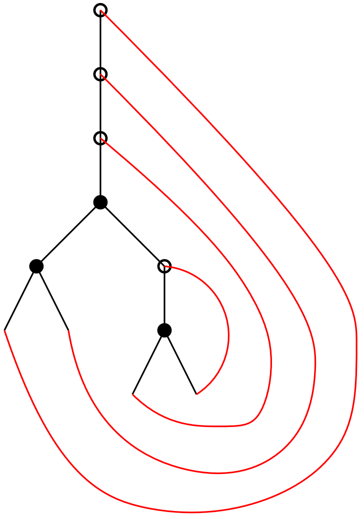

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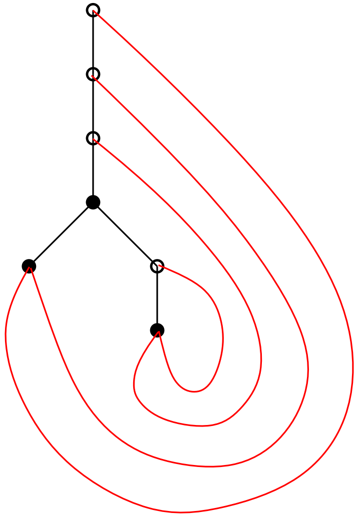

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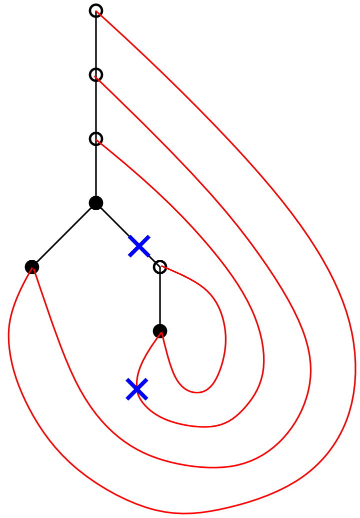

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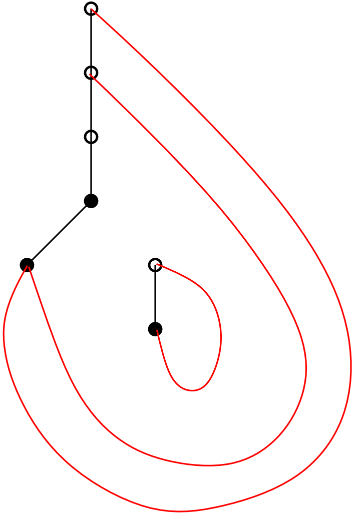
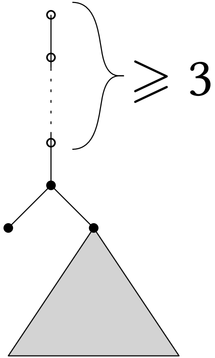


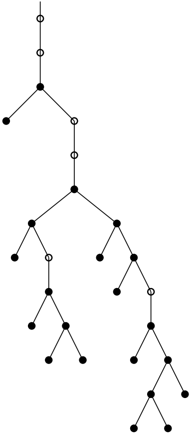
Properties

- Every term of size at least 3 has at least three head lambdas.
- The leftmost maximal binary subtree is just a single leaf.



Properties

- In every non-trivial maximal unary-binary subtree starting with a unary node the right subtree rooted at the highest binary node has more leaves than the number of its unary nodes increased by the number of head unary nodes.



# 'right' unary nodes	# 'right' leaves

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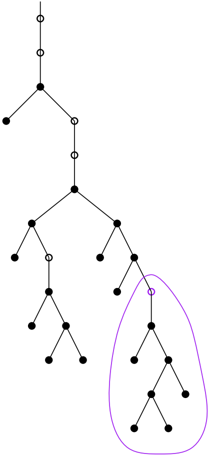

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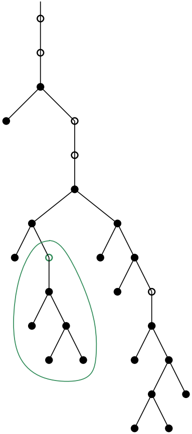

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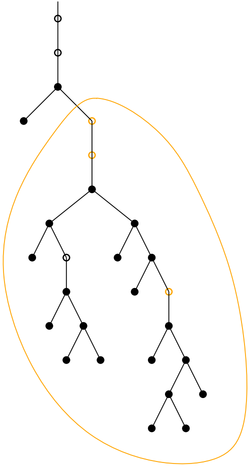
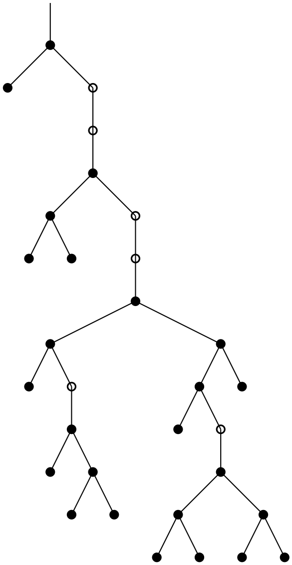


# 'right' unary nodes	# 'right' leaves
3	6

Sending gifts and skipping top unary nodes

Without any loss of information, we perform two operations on the unary-binary trees of our interest.

- Skipping all head unary nodes.
- Sending gifts (as depicted).



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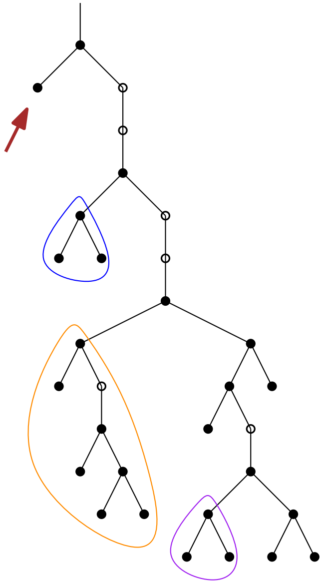

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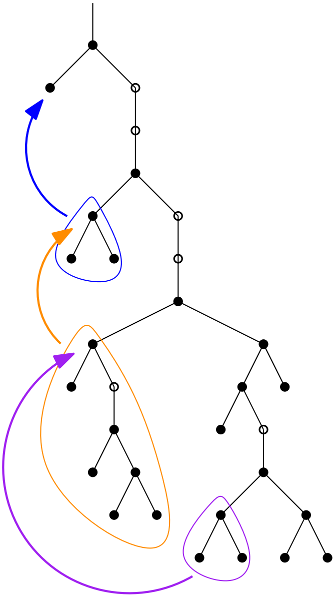

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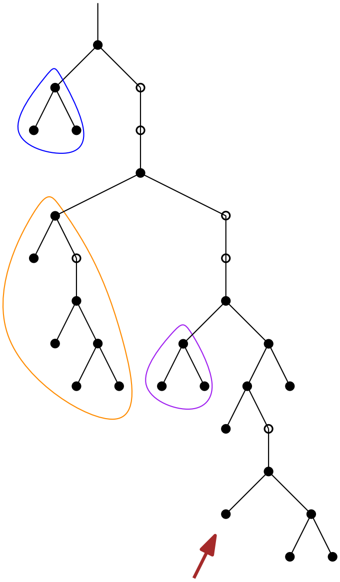
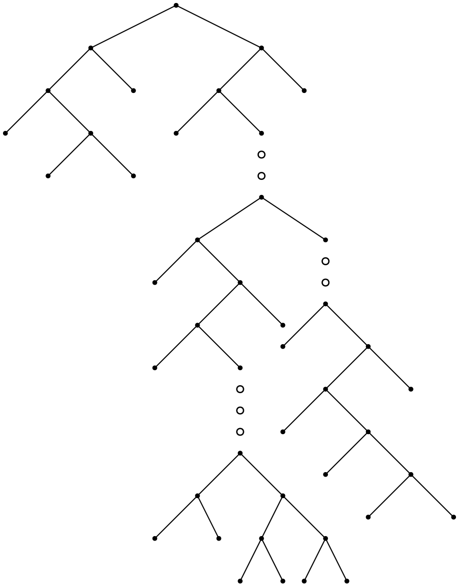


Decomposition of trees after pre-processing



Bicubic maps and $\beta(0,1)$-trees

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A **$\beta(0, 1)$-tree** is a rooted plane tree whose nodes are labeled with non-negative integers in the following way:

- leaves have label 0;
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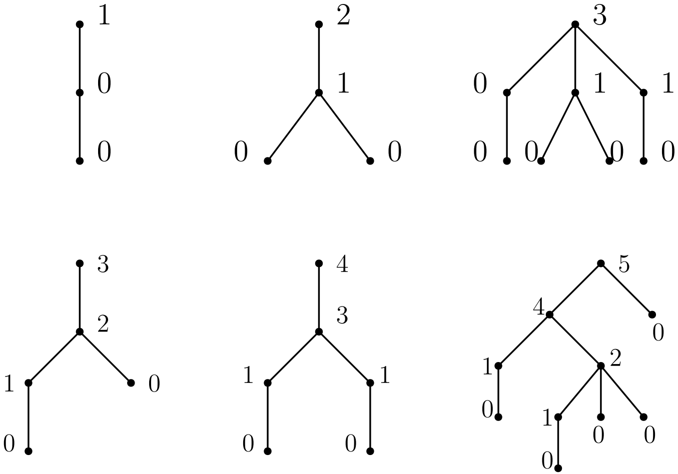
So let's play with $\beta(0,1)$-trees!

Examples of $\beta(0,1)$-trees



$\beta(0,1)$-bricks

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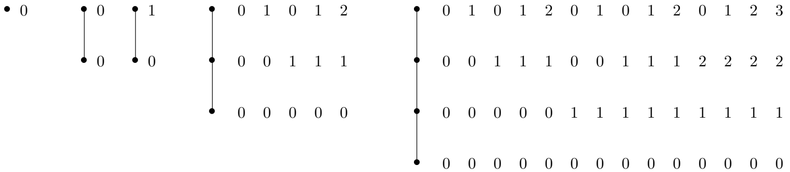

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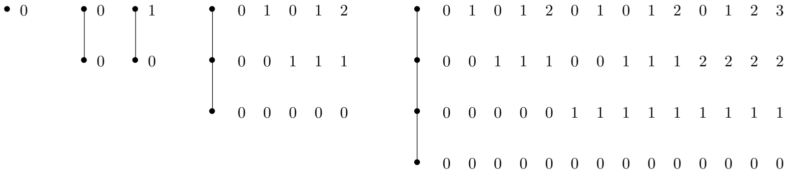

Bricks are enumerated by Catalan numbers.

Labelling binary trees

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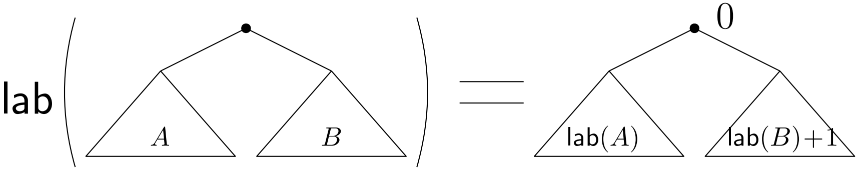

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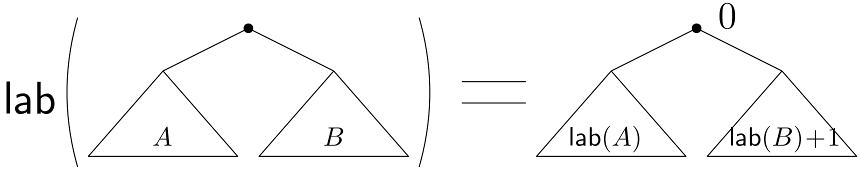
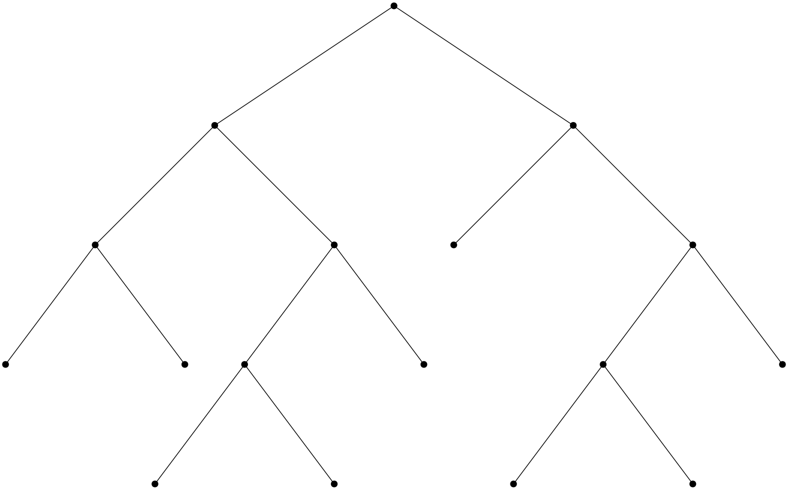
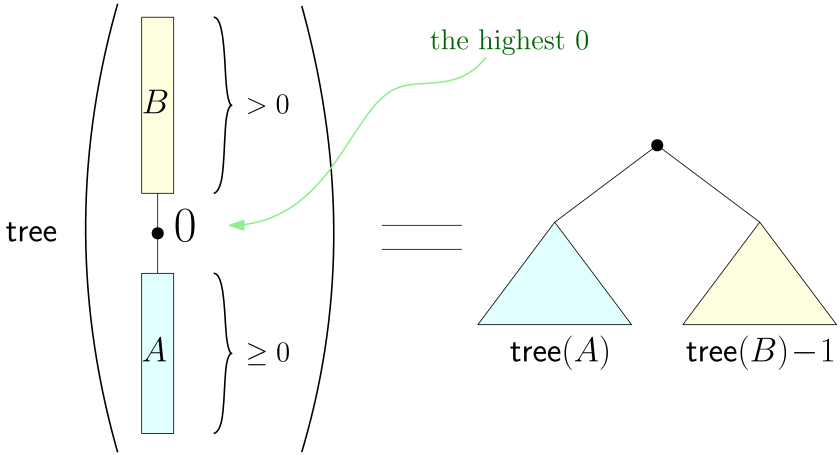


We use the notions ‘binary tree’ and ‘labelled binary tree’ interchangeably.

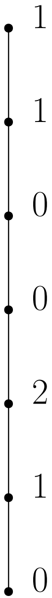
Labelling binary trees



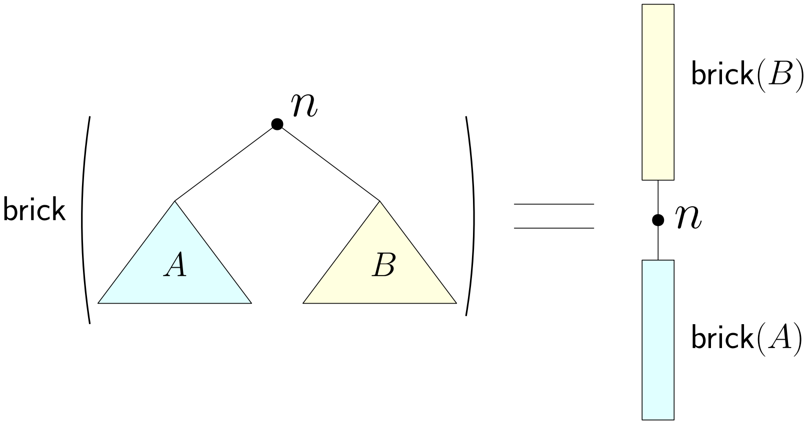
Bricks to trees



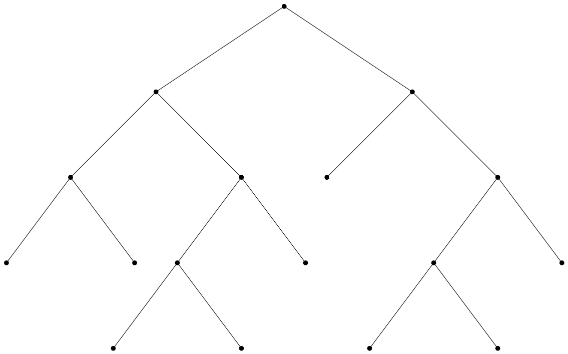
Bricks to trees



Trees to bricks



Trees to bricks



The ultimate goal

- $\beta(0,1)$-trees $\longrightarrow$ (several adjectives) lambda terms

  How to decompose $\beta(0,1)$-trees into bricks and how to combine the corresponding binary trees together in order to obtain lambda terms?

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  How to encode bricks corresponding to the binary trees from the decomposition of lambda terms and how to glue them together in order to obtain $\beta(0,1)$-trees?

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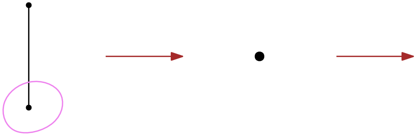
  How to encode bricks corresponding to the binary trees from the decomposition of lambda terms and how to glue them together in order to obtain $\beta(0,1)$-trees?

In the case of our candidate for a bijection…

…the number of maximal binary trees in the decomposition of lambda trees corresponds to the number of leaves in $\beta(0,1)$-trees.

Translation: trivial cases

$\lambda x.x$



empty $\beta(0,1)$-tree

Translation: trivial cases

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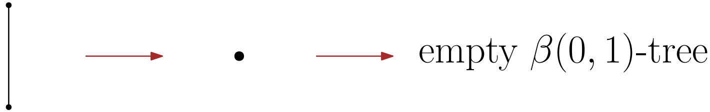

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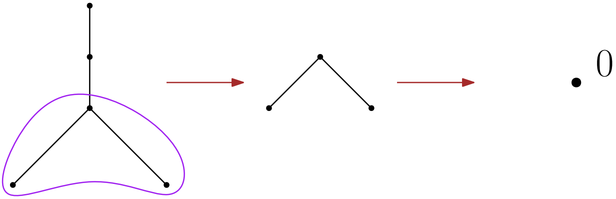
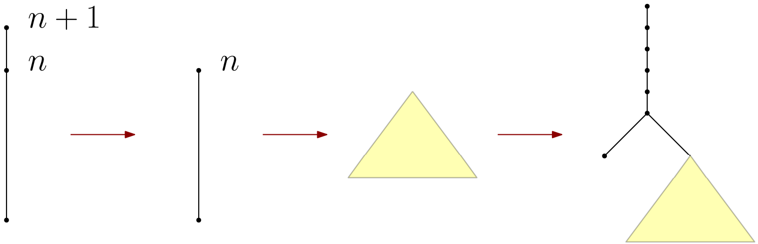


Translation: β(0, 1)-trees with one leaf



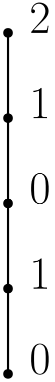
$\beta(0,1)$-tree with 1 leaf

brick

binary tree

lambda term with 1 part

Example: $\beta(0,1)$-trees with one leaf



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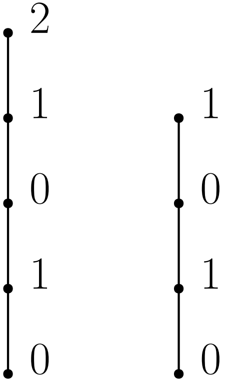

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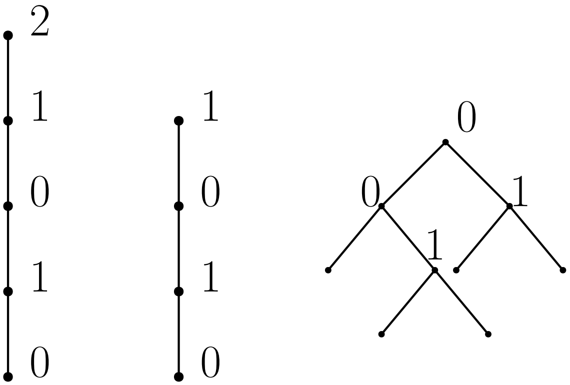

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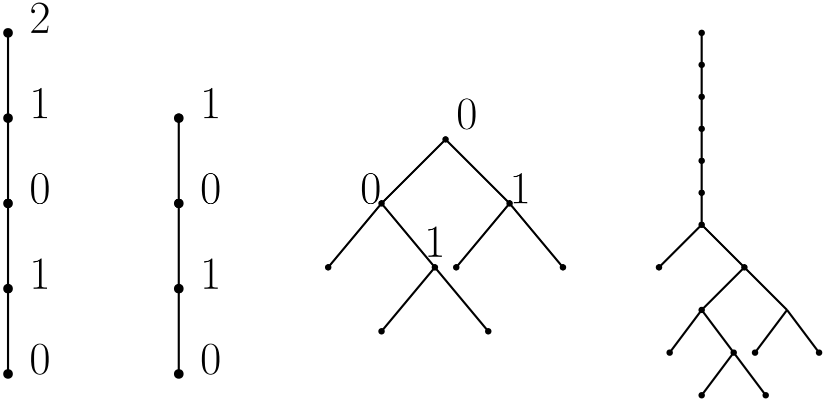
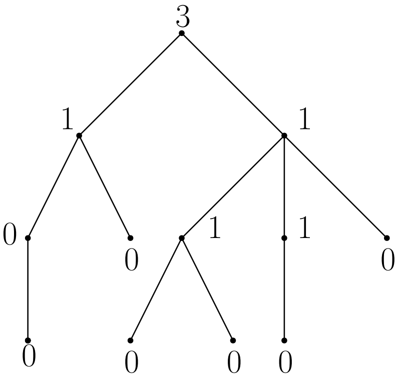


Translation: $\beta(0,1)$-trees with no jumps



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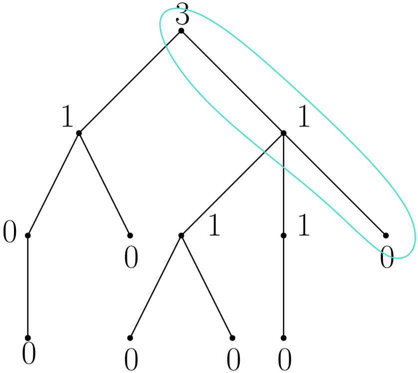

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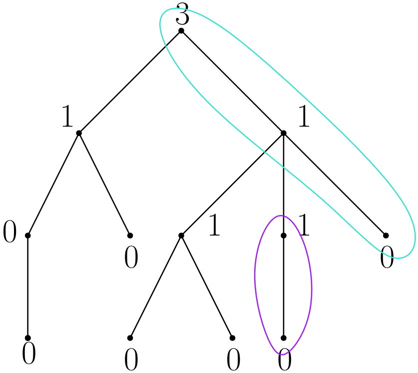

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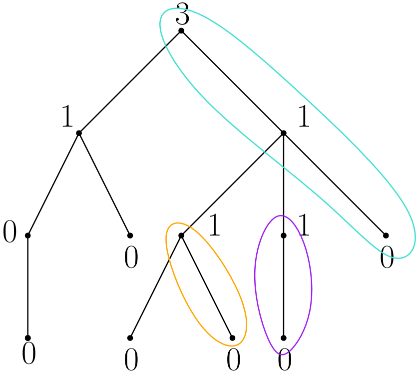

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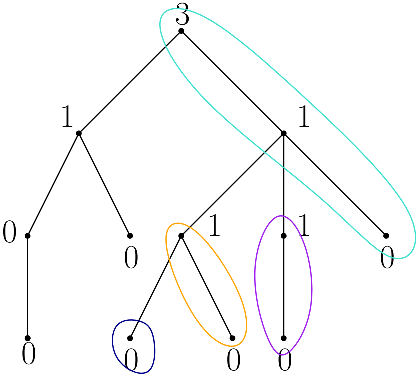

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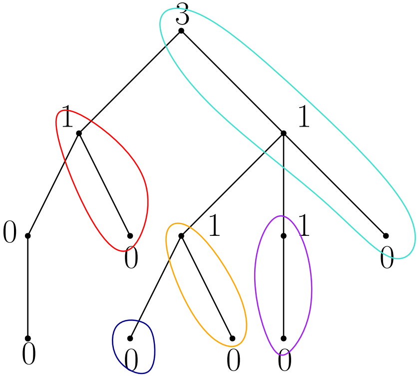

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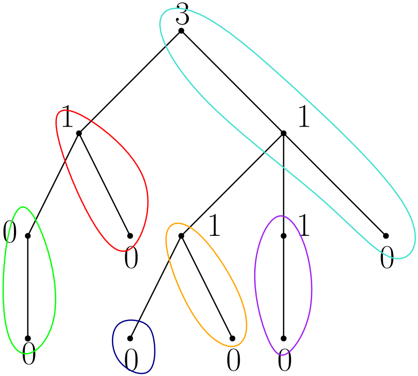

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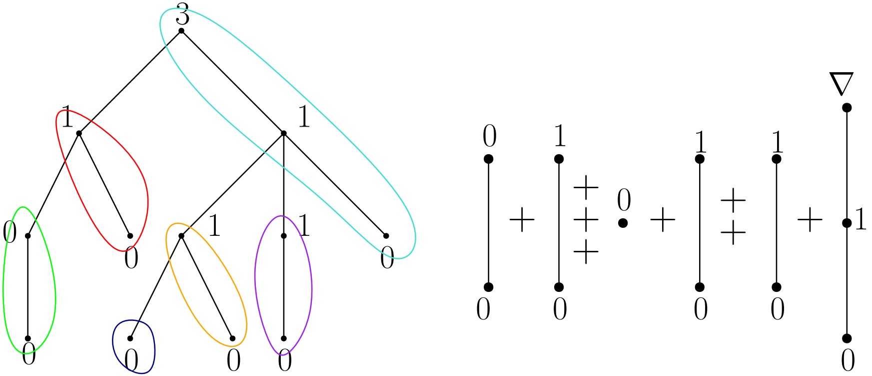

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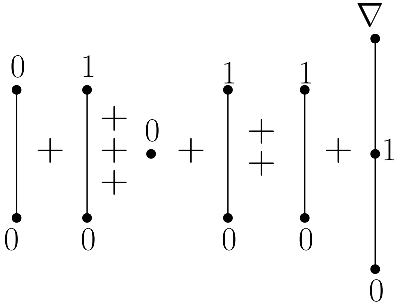

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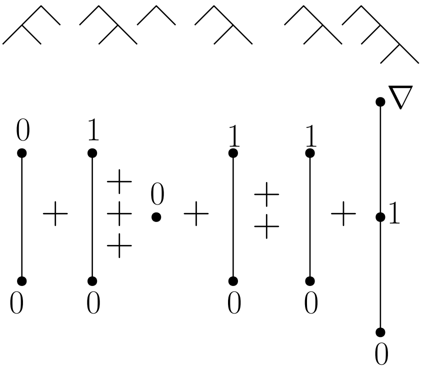

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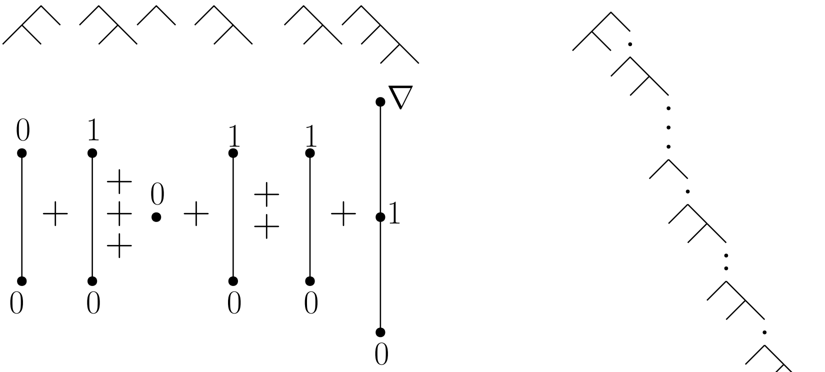

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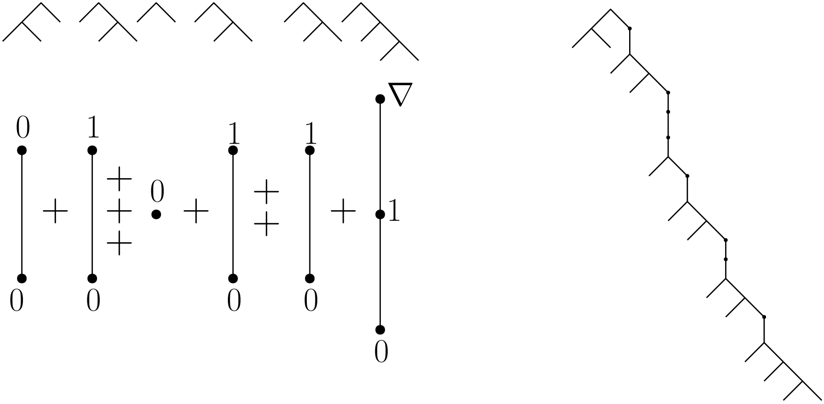

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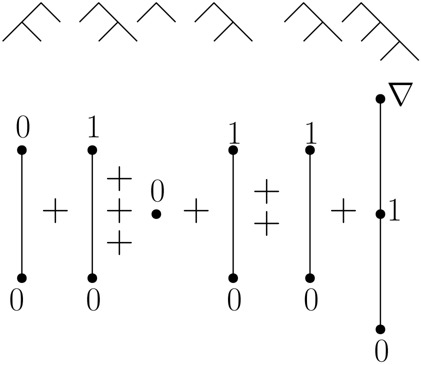

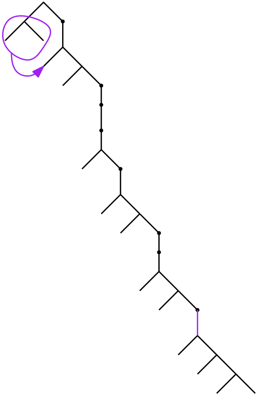

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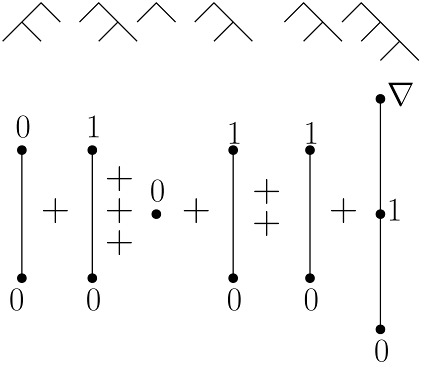

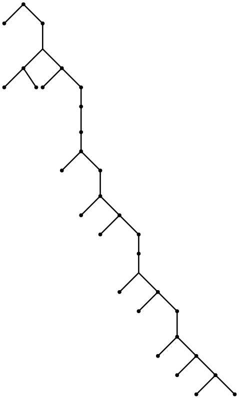

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An internal node in a $\beta(0, 1)$-tree is a **jump** iff

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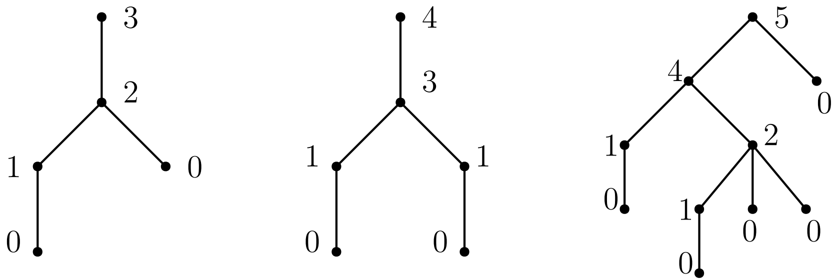

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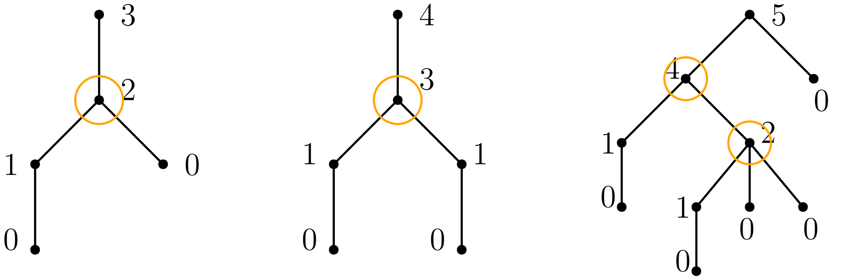
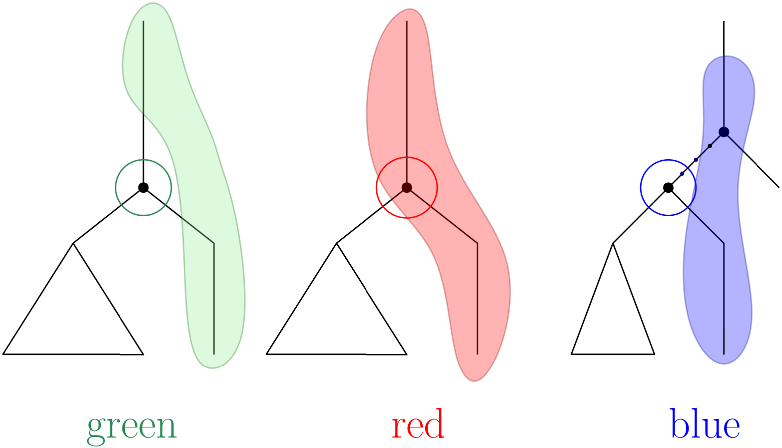
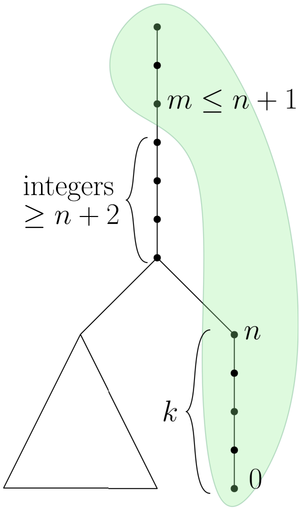


RGB jumps

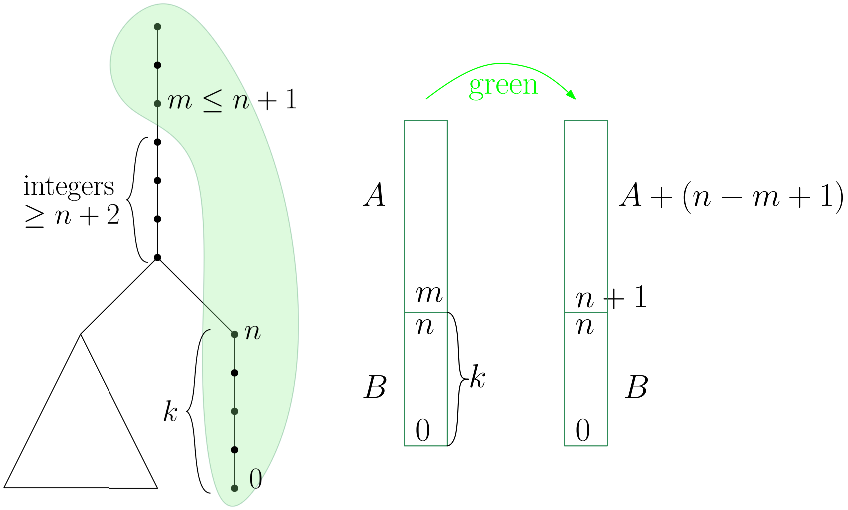
We distinguish three kinds of jumps related to the character of decomposition of $\beta(0, 1)$-trees.



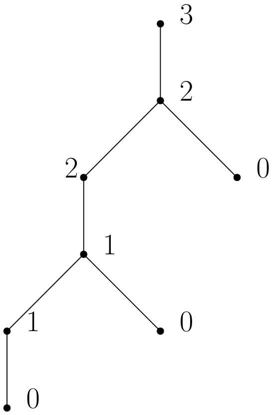
Green jump and green transformation



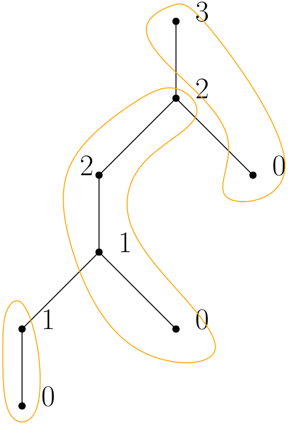
Green jump and green transformation



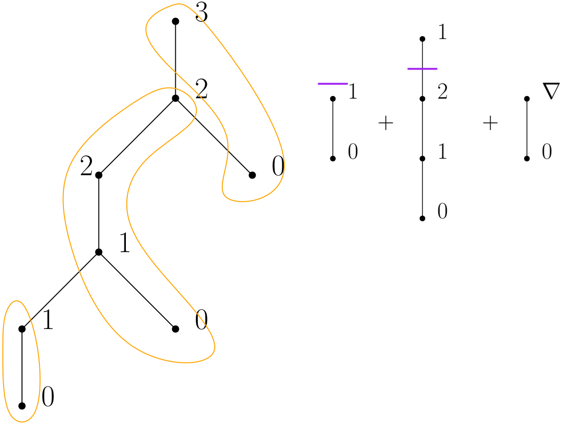
Green example 1



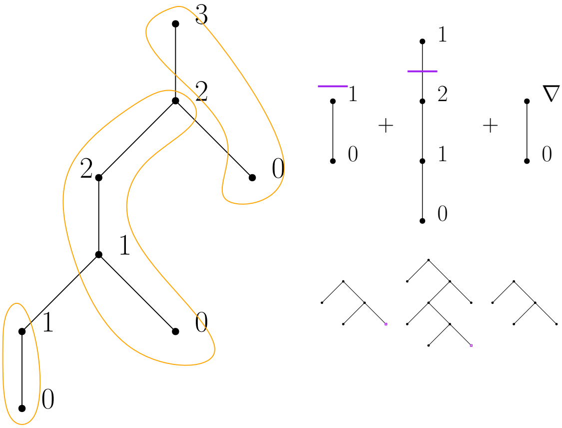
Green example 1



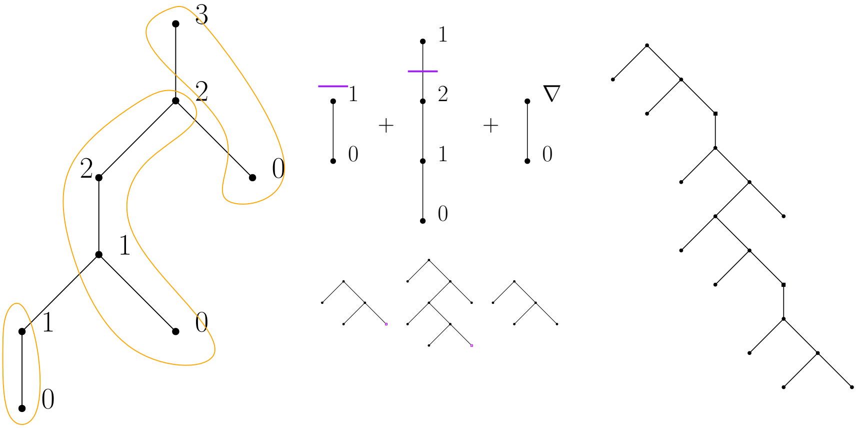
Green example 1



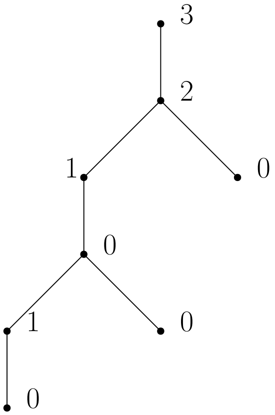
Green example 1



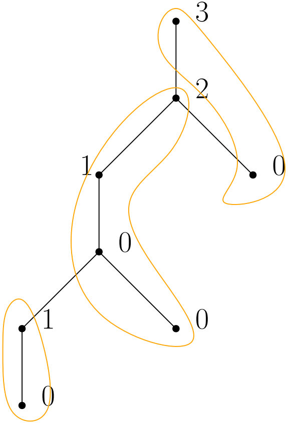
Green example 1



Green example 2



Green example 2



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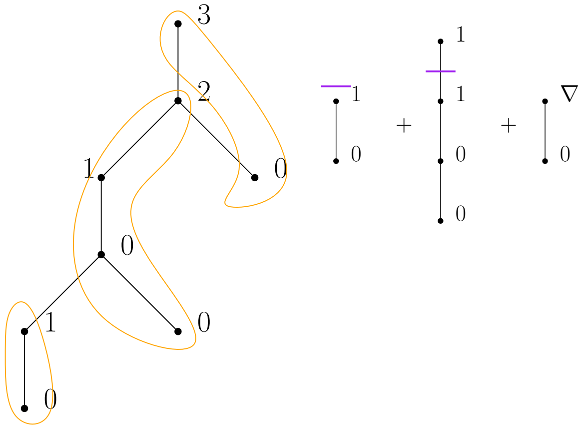

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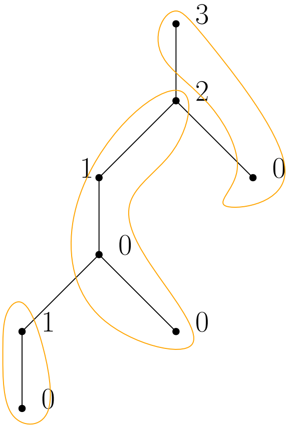

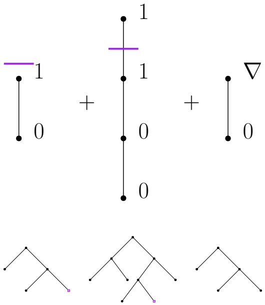

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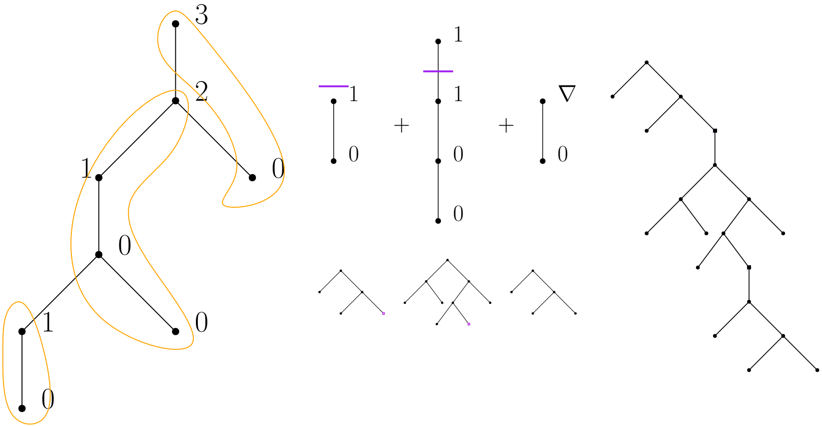
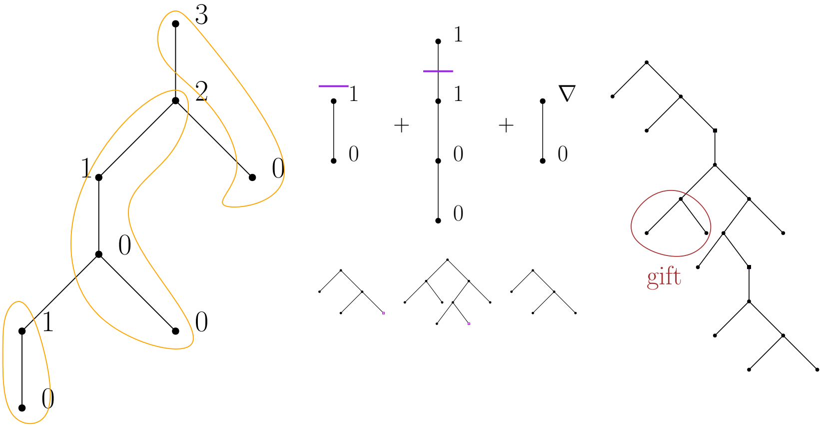
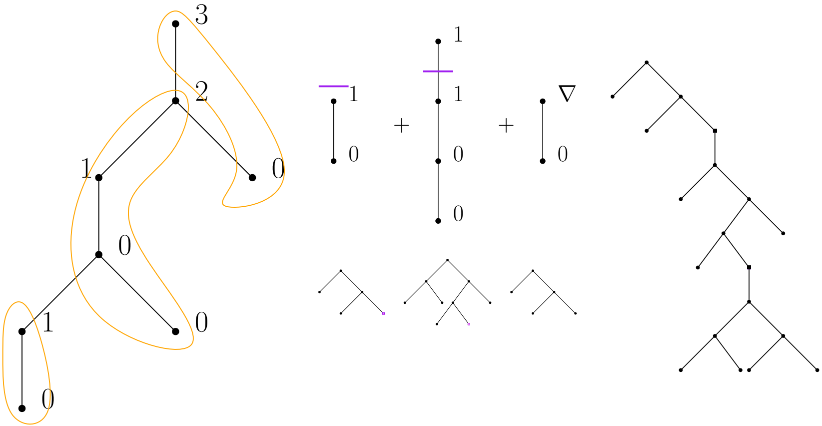


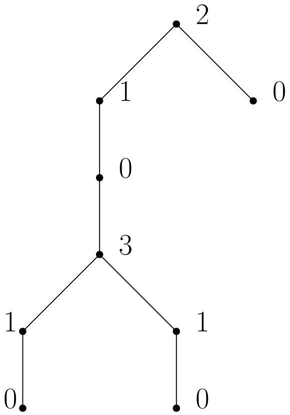
Green example 2



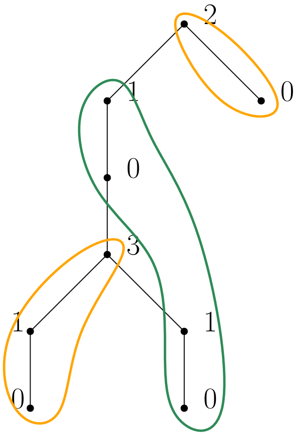
Green example 2



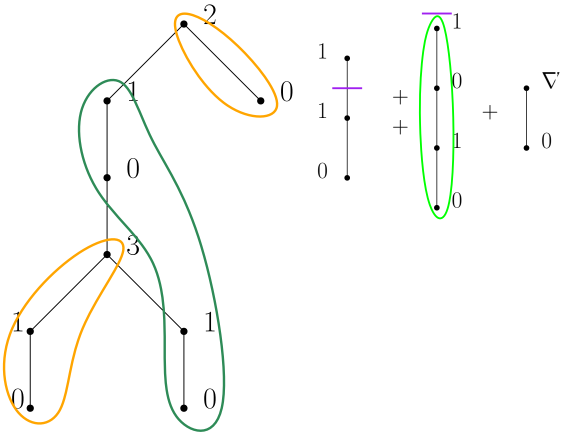
Green example 3



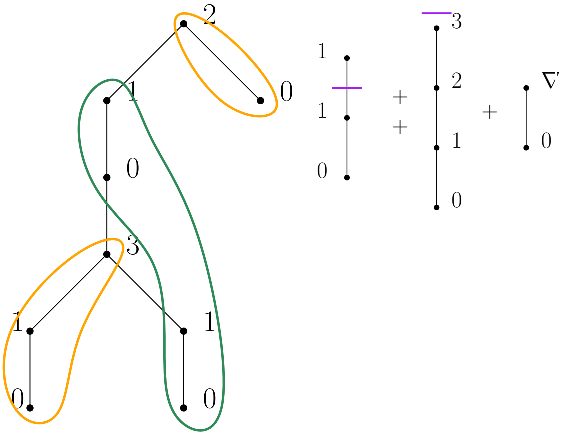
Green example 3



Green example 3



Green example 3



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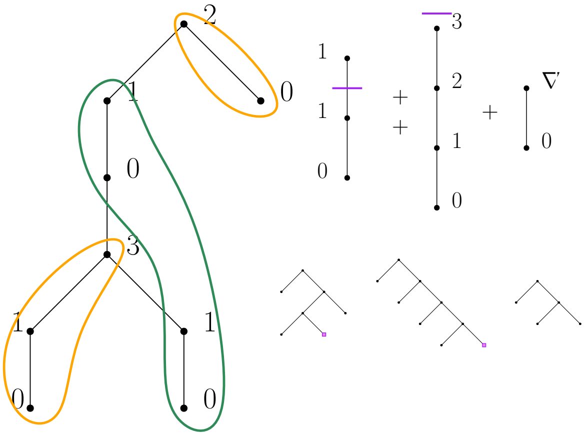

Green example 3