A note on the asymptotic expressiveness of ZF and ZFC

Maciej Bendkowski

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Problem statement

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Definition

We define the *asymptotic expressiveness* of a theory $\mathcal{T}$ as the asymptotic density of its theorems among all possible sentences:

$$\mu(\mathcal{T}) = \lim_{n\to\infty} \frac{|\{\varphi\colon |\varphi| = n \wedge \mathcal{T} \vdash \varphi\}|}{|\{\varphi\colon |\varphi| = n\}|}. \tag{1}$$

Formulae

The language of Zermelo-Fraenkel set theory consists of a single binary *membership* predicate ($\in$) and no function symbols. More specifically, assume that $\mathcal{F}$ is a functionally complete set of proposition connectives, *e.g.* $\{\wedge, \vee, \neg\}$. Then,

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Example

$\exists x.\ [\forall y.\,(y \notin x)].$

Formulae

In the current work, we adopt the De Bruijn notation for formulae.

Example

$\exists x.\ [\forall y. (y \notin x)] \equiv \exists\ [\forall (\underline{0} \notin \underline{1})]$.

Symbolically, the set Φ_∞ of formulae is defined as

$$\Phi_\infty = \ V \in V \mid \forall \Phi_\infty \mid \exists \Phi_\infty \mid \bigcup_{(\circ) \in \mathcal{F}} \circ(\vec{\Phi}_\infty)$$

where indices V are represented in unary, *i.e.* $\underline{\mathrm{n}} = S^{(n)}0$.
Consequently V is specified by $V = 0 \mid SV$.

Size model

We assume a *natural* size notion for formulae. In other words, the *size* $|\varphi|$ of a formula φ is the number of its building constructors.

Example

$|\exists\ [\forall\,(\underline{0} \notin \underline{1})]| = 7.$

$$\Phi_\infty = V \in V \mid \forall\Phi_\infty \mid \exists\Phi_\infty \mid \bigcup_{(\circ)\in\mathcal{F}} \circ(\vec{\Phi}_\infty)$$

Therefore

$$\Phi_\infty(z) = z\left(\frac{z}{1-z}\right)^2 + 2z\Phi_\infty(z) + \sum_{(\circ)\in\mathcal{F}} z\Phi_\infty(z)^{\mathsf{arity}(\circ)}.$$

Counting formulae

Proposition

The generating function Φ_∞ admits a Puiseux expansion in form of

$$\Phi_\infty(z) = a - b\sqrt{1 - \frac{z}{\rho}} + O\left(\left|1 - \frac{z}{\rho}\right|\right)$$

Moreover

$$[z^n]\Phi_\infty(z) \sim C \cdot \rho^n n^{-3/2}.$$

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Proof.

By Post's theorem $\mathcal{F}$ must contain a connective of arity $k \geq 2$. In consequence the defining equation of Φ_∞ is non-linear in Φ_∞. We can therefore apply the Drmota–Lalley–Woods theorem. □

Counting sentences

Recall that we are interested in *sentences* rather than formulae. Hence we want to count formulae in which each index is *bound*, *i.e.* has a corresponding quantifier.

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Definition

We call a formula φ *m-open* if prepending φ with m head quantifiers, be it universal or existential, turns φ into a sentence, *i.e.* a formula without free indices.

Note that sentences are 0-open (closed) formulae.

Counting sentences

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$$\Phi_m := \forall\Phi_{m+1} \mid \exists\Phi_{m+1} \mid \bigcup_{(\circ)\in\mathcal{F}} \circ(\vec{\Phi}_m) \mid V_{<m} \in V_{<m}$$

$$\dots$$

Counting sentences

Note that $\Phi_0 \subsetneq \Phi_1 \subsetneq \Phi_2 \subsetneq \cdots \subsetneq \Phi_m \cdots \subsetneq \Phi_\infty$ and so, intuitively,

$$\Phi_m \xrightarrow[m\to\infty]{} \Phi_\infty.$$

Proposition

For all $m \geq 0$, the number of m-open formulae satisfies

$$[z^n]\Phi_m(z) \sim C_m \cdot \rho^n n^{-3/2}.$$

Idea: Re-use tools developed to count closed λ-terms[1]. Note that

$$\Phi_m(z) = z\left(\frac{z\,(1-z^m)}{1-z}\right)^2 + 2z\Phi_{m+1}(z) + \sum_{(\circ)\in\mathcal{F}} z\Phi_m(z)^{\text{arity}(\circ)}$$

$$\Phi_\infty(z) = z\left(\frac{z}{1-z}\right)^2 + 2z\Phi_\infty(z) + \sum_{(\circ)\in\mathcal{F}} z\Phi_\infty(z)^{\text{arity}(\circ)}.$$

[1] B., Bodini, Dovgal. *"Statistical properties of lambda-terms". EJC. 2019.*

Counting theorems

Plan

Since we can now estimate $[z^n]\Phi_0(z)$, *i.e.* the number of sentences of size n, we need to focus on the number of *theorems* of size n.

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Problem

Deciding whether a given sentence φ is a theorem of ZF (or ZFC) is *undecidable*. So is to compute $|\{\varphi \colon |\varphi| = n \wedge \text{ZF} \vdash \varphi\}|$.

Interlude — a few intuitions

What does it mean that $\mathcal{T} \vdash \varphi$?

1. there exists a *proof tree* whose leaves are (instantiations of) axioms of $\mathcal{T}$ or predicate logic, internal nodes represent valid *inference rules* (*e.g. modus ponens*), and the root represents φ.

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1. $I(f): U^n \to U$ if f is an n-ary symbol.
2. $I(P): U^n \to \{\text{false}, \text{true}\}$ if P is n-ary predicate.

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Definition

A theory $\mathcal{T}$ is said be *consistent* if $\mathcal{T} \nvdash \varphi \wedge \neg\varphi$ for any sentence φ or, equivalently, if $\mathcal{T}$ has a model. $\mathcal{T}$ is said to be *complete* if for all sentences φ it holds $\mathcal{T} \vdash \varphi$ or $\mathcal{T} \vdash \neg\varphi$.

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Definition

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Gödel's incompleteness theorems

Sufficiently expressive theories (such as ZF or ZFC) cannot be both consistent and complete. Moreover, *sufficiently expressive* and consistent theories cannot prove their own consistency.

Formulae templates

Definition

A *template* $\mathcal{C}$ is a formula with a single hole $[\cdot]$ instead of some sub-formula in form of $(\underline{\mathrm{n}} \in \underline{\mathrm{m}})$. To denote the result of substituting a formula φ for $[\cdot]$ in $\mathcal{C}$ we write $\mathcal{C}[\varphi]$.

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Definition

We call a template *m-permissive* if for each *m*-open formula φ the resulting $\mathcal{C}[\varphi]$ is a sentence, *i.e.* is 0-open. By analogy to formulae, the *size* of a template is the total weight of its building constructors, assuming that $[\cdot]$ weights zero.

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Example

The template $\mathcal{C} = \exists\, \forall\, [\cdot]$ is of size two. The result of substituting $(\underline{0} \notin \underline{1})$ into $\mathcal{C}$ is the formula $\mathcal{C}[(\underline{0} \notin \underline{1})] = \exists\, \forall\, (\underline{0} \notin \underline{1})$. The hole $[\cdot]$ is proceeded by two quantifiers in $\mathcal{C}$ so $\mathcal{C}$ is 2-permissive.

Results

Lemma

Let $\mathcal{C}$ be an *m*-permissive template and $\mathcal{L}(\mathcal{C})$ be the language it generates, *i.e.* $\mathcal{L}(\mathcal{C}) = \{\mathcal{C}[\varphi] \colon \varphi \text{ is } m\text{-open}\}$. Then, the set $\mathcal{L}(\mathcal{C})$ has positive asymptotic density in the set of all sentences.

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Proposition

Let $\mathcal{T}$ be a consistent set theory. Then, the set of $\mathcal{T}$-theorems cannot have a trivial asymptotic density, *i.e.* neither $\mu(\mathcal{T}) \neq 0$ nor $\mu(\mathcal{T}) \neq 1$. Moreover, $\mathcal{T}$ is inconsistent if and only if $\mu(\mathcal{T}) = 1$.

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Proof.

Let τ be an arbitrary tautology. Consider $\mathcal{C} = ([\cdot] \vee \tau)$ and $\overline{\mathcal{C}} = ([\cdot] \wedge \neg\tau)$. Note that these have positive asymptotic density and consist of tautologies and anti-tautologies, respectively. □

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Proposition

Let $\mathcal{T}$ be a consistent theory and ϕ be an sentence independent of $\mathcal{T}$, *i.e.* $\mathcal{T} \not\vdash \phi$ nor $\mathcal{T} \not\vdash \neg\phi$. Then, there exists a set of theorems of the extended $\mathcal{T} + \phi$ which has positive asymptotic density.

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Proof.

Let τ be an arbitrary tautology. Consider the context $\mathcal{C} = (\tau \vee [\cdot]) \rightarrow \phi$ and the set $\mathcal{L}(\mathcal{C}) = \{\mathcal{C}[\varphi] \colon \varphi \in \Phi_0\}$ it generates. Note that $\mathcal{L}(\mathcal{C})$ has positive asymptotic density and consists of sentences which cannot be proven in the weaker theory $\mathcal{T}$. □

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Corollary

The axiom of choice is an example of a sentence independent of ZF hence ZFC and ZF cannot share the same asymptotic expressiveness.

Results

Proposition

Let μ be a predicate definable in ZFC such that ZFC $\vdash \mu(g)$ if and only if $\mu($ZFC$)$ exists and is equal to g. Let CONSISTENT be the (canonical) predicate encoding the consistency of ZFC. If ZFC is consistent and

$$\text{ZFC} \vdash \text{CONSISTENT} \longleftrightarrow \neg\mu(1), \tag{2}$$

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Proof.

Suppose that ZFC $\vdash \neg\exists\, g\colon \mu(g)$. Equivalently, ZFC $\vdash \forall\, g\colon \neg\mu(g)$. Hence in particular ZFC $\vdash \neg\mu(1)$. By (2) it holds ZFC $\vdash$ CONSISTENT. Gödel's second incompleteness theorem, as instantiated for ZFC, provides the required contradiction. □

Conclusions and remarks

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- $\mathcal{M} \vDash \exists\, g\colon\, \mu(g)$ holds for each model $\mathcal{M}$ of ZFC. *A priori* these witnesses do not have to be the same across all models.

Conclusions and remarks

Thank you!