A NOTE ON THE ASYMPTOTIC EXPRESSIVENESS OF ZF AND ZFC

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Abstract. We investigate the asymptotic densities of theorems provable in Zermelo-Fraenkel set theory ZF and its extension ZFC including the axiom of choice. Assuming a canonical De Bruijn representation of formulae, we construct asymptotically large sets of sentences unprovable within ZF, yet provable in ZFC. Furthermore, we link the asymptotic density of ZFC theorems with the provable consistency of ZFC itself. Consequently, if ZFC is consistent, it is not possible to refute the existence of the asymptotic density of ZFC theorems within ZFC. Both these results address a recent question by Zaionc regarding the asymptotic equivalence of ZF and ZFC.

1. Introduction

In the current paper we are interested in the *asymptotic expressiveness* of first-order set theories ZF and ZFC. More specifically, we investigate the asymptotic density of sentences provable within these theories among all sentences expressible in the first-order language $\mathcal{L}$ consisting of a single binary *membership* predicate $(\in)$ and no function symbols.

We start with the following problem posted recently by Zaionc.

Problem. Consider the theories ZF and ZFC. What is the asymptotic density of theorems provable within ZFC? Is it true that ZFC is asymptotically more expressive than ZF?

To make the notion of asymptotic density of theorems sound, we have to assign to each formula φ an integer *size* $|\varphi|$ in such a way that there exists a finite number of formulae of any given size. Having such a size notion, we then define the *asymptotic expressiveness* of a theory $\mathcal{T}$ as the *asymptotic density* $\mu(\mathcal{T})$ of its theorems among all possible sentences, *i.e.*

$$\mu(\mathcal{T}) = \lim_{n\to\infty} \frac{|\{\varphi\colon |\varphi| = n \land \mathcal{T} \vdash \varphi\}|}{|\{\varphi\colon |\varphi| = n\}|}. \tag{1}$$

In order to start addressing the above problem, we have to establish a formal framework in which we fix certain technical, yet important details, such as the assumed size model of formulae, or their specific combinatorial representation. In this paper we choose to represent formulae using De Bruijn indices [de 72] instead of the usual notation involving named variables. Within this setup our contributions are twofold.

Firstly, we show that ZF and ZFC cannot share the same asymptotic expressiveness. Specifically, we construct an asymptotically large (*i.e.* having positive asymptotic density) fraction of $\mathcal{L}$-sentences which, though provable in ZFC, cannot be proven in the weaker system ZF without the axiom of choice. Secondly, we show that it is not possible to refute the existence of $\mu(\text{ZFC})$ within ZFC itself. For that purpose we link the provable existence of $\mu(\text{ZFC})$ with the provable consistency of ZFC. In light of Gödel's second incompleteness theorem, the existence of $\mu(\text{ZFC})$ becomes unprovable within ZFC.

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We base our analysis on a mixture of methods from analytic combinatorics and, more specifically, recent advances in its application in the quantitative analysis of λ-terms [BBD19; BGG18; FS09].

References

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