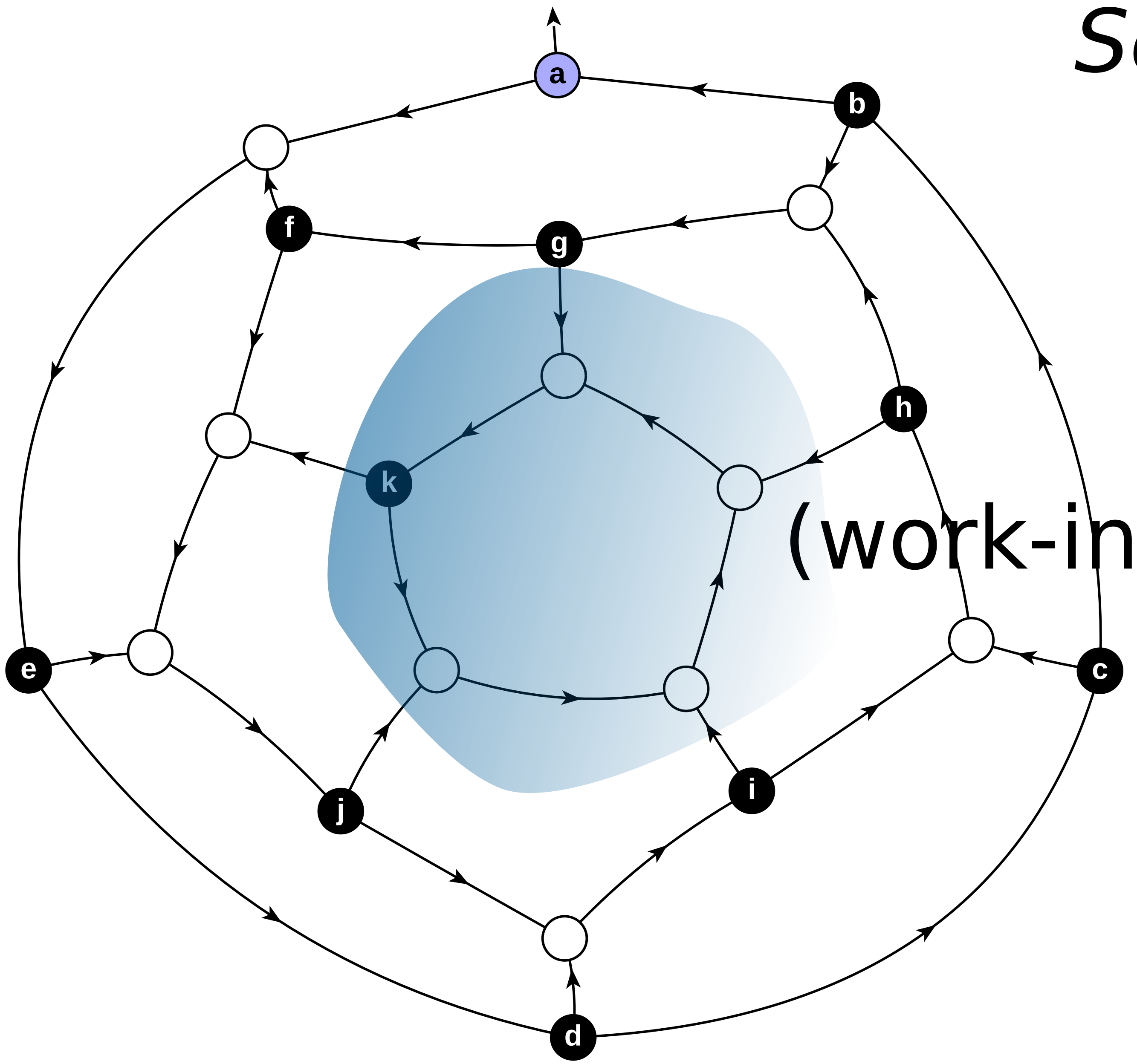


Some topological properties of planar lambda terms

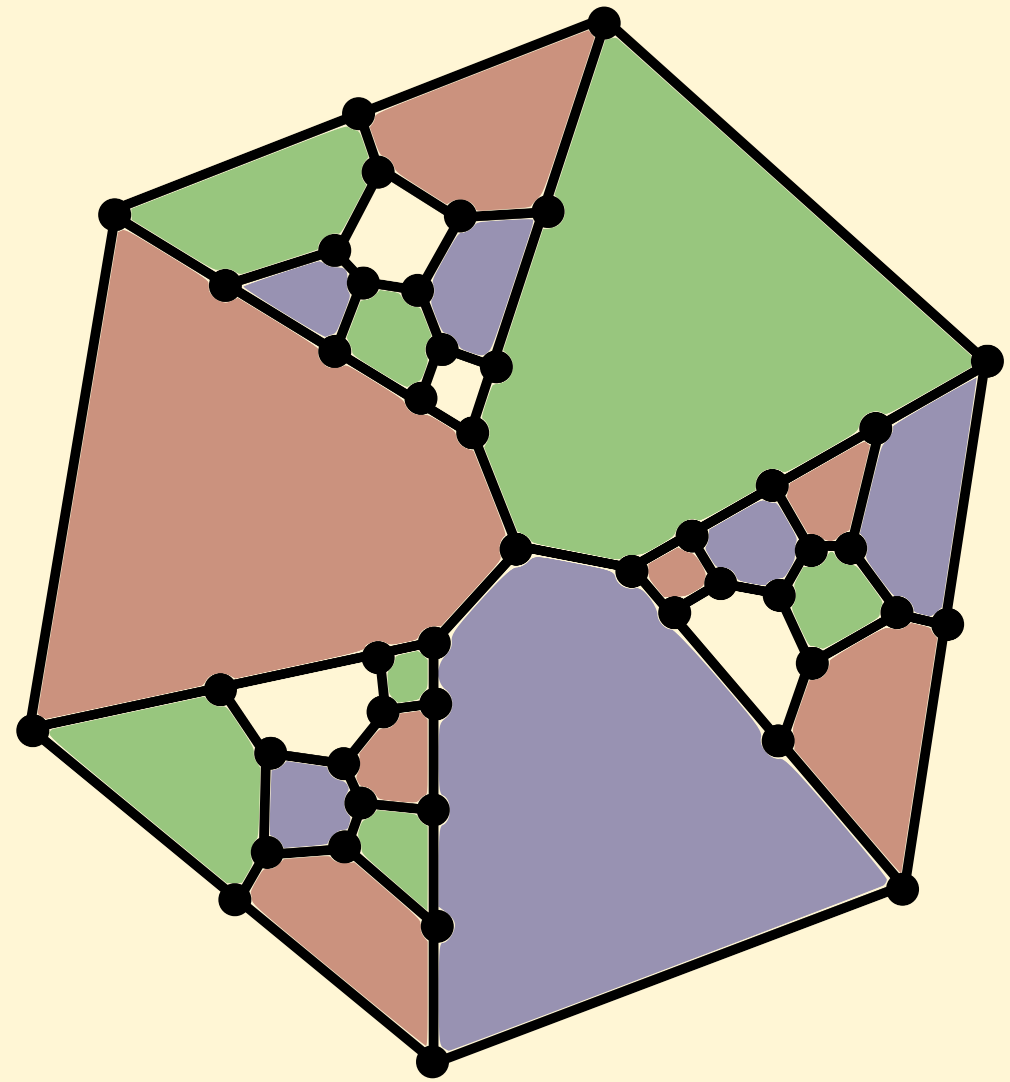
Noam Zeilberger

(work-in-progress with Jason Reed)

CLA 2019
Versailles, 1-2 July

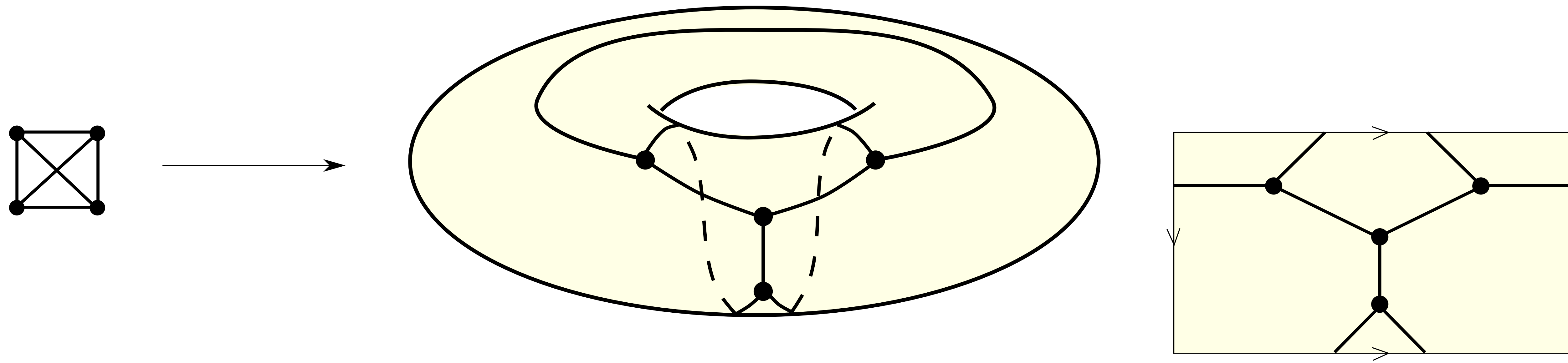
[Background]

A few views on maps



Topological definition

map = 2-cell embedding of a graph into a surface*



considered up to deformation of the underlying surface.

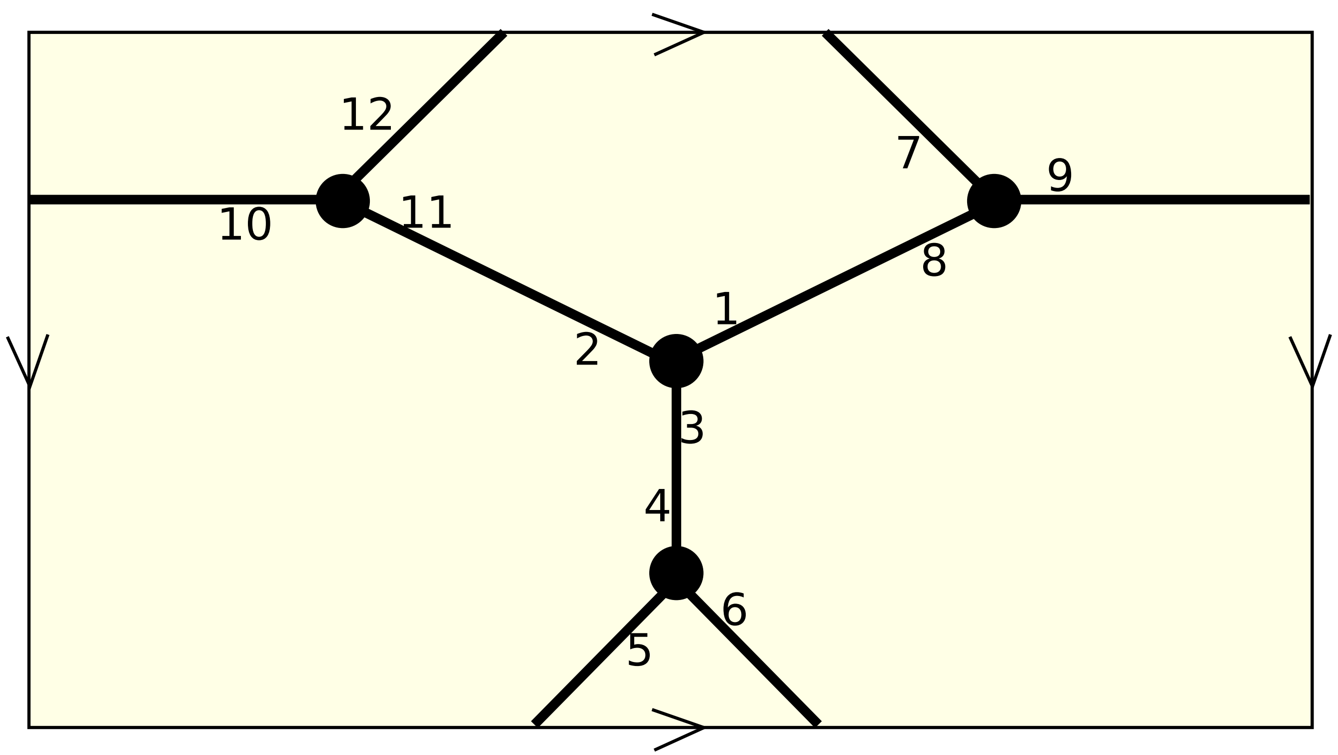
*All surfaces are assumed to be connected and oriented throughout this talk

Algebraic definition

map = transitive permutation representation of the group

G = $\langle v, e, f \mid e^2 = vef = 1 \rangle$

considered up to G-equivariant isomorphism.



$$v = (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12)$$

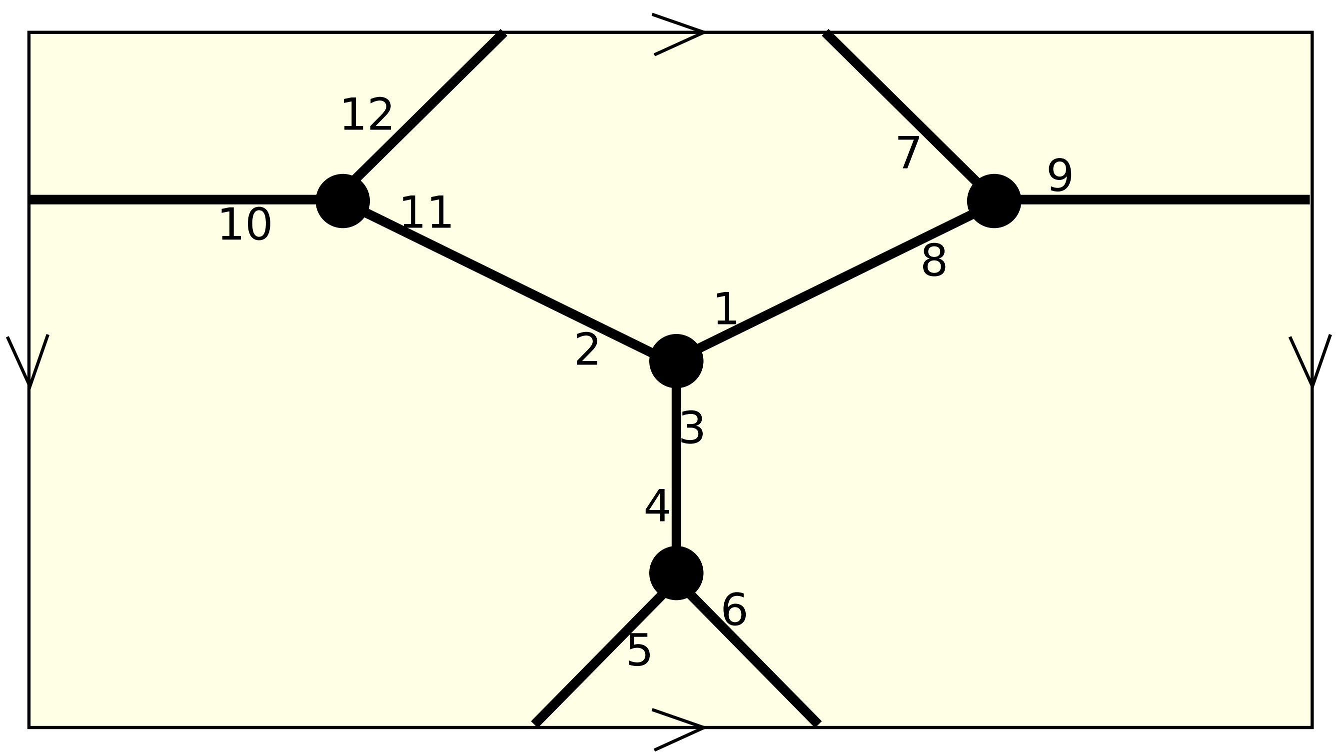
$$e = (1\ 8)(2\ 11)(3\ 4)(5\ 12)(6\ 7)(9\ 10)$$

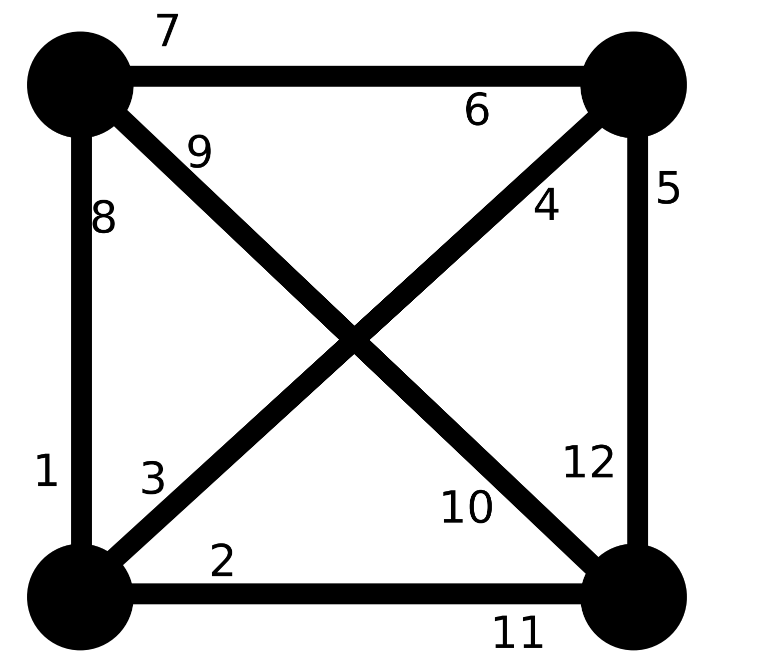
$$f = (1\ 7\ 5\ 11)(2\ 10\ 8\ 3\ 6\ 9\ 12\ 4)$$

$$c(v) - c(e) + c(f) = 2 - 2g$$

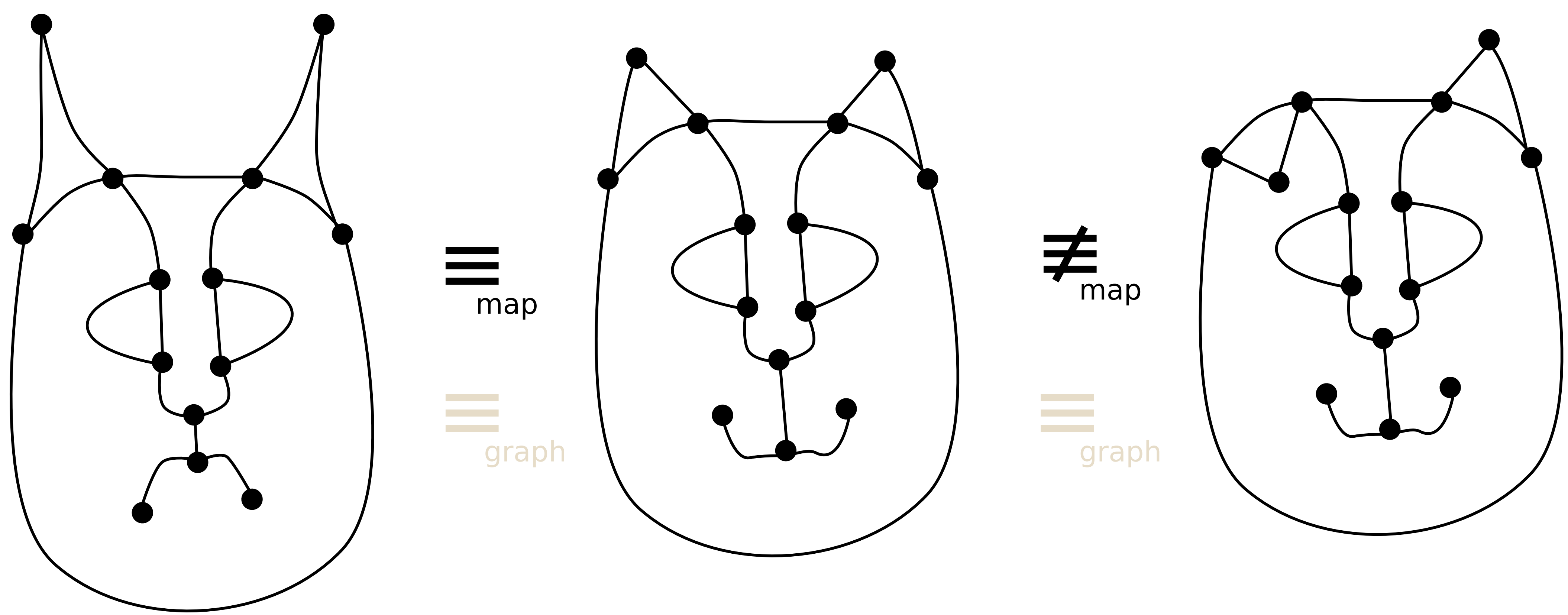
Combinatorial definition

map = connected graph + cyclic ordering of the half-edges around each vertex (say, as given by a drawing with "virtual crossings").

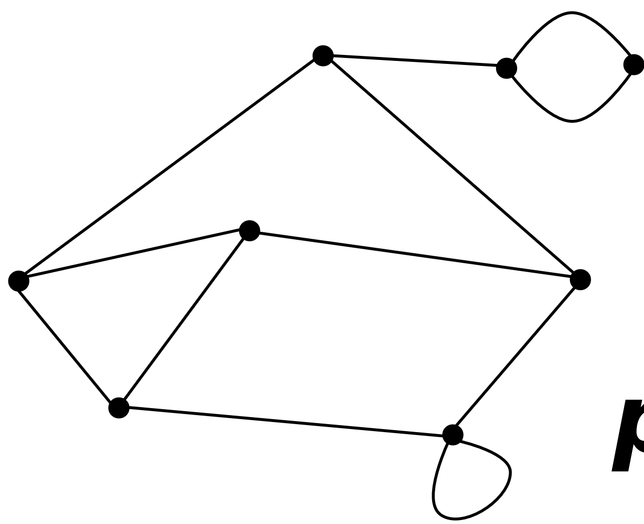




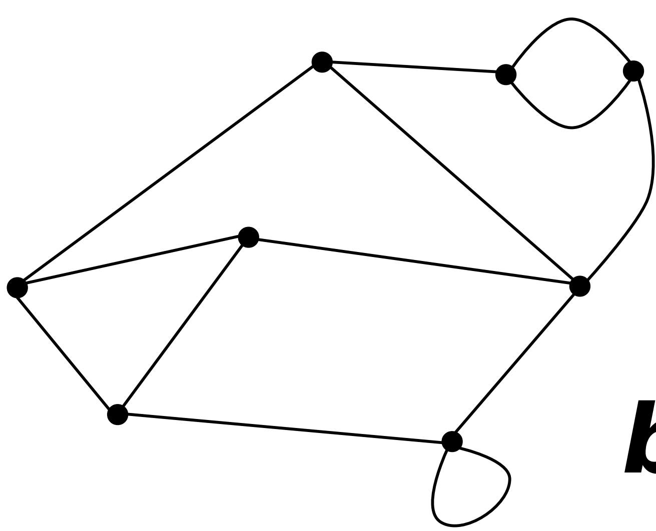
Graph versus Map



Some special kinds of maps

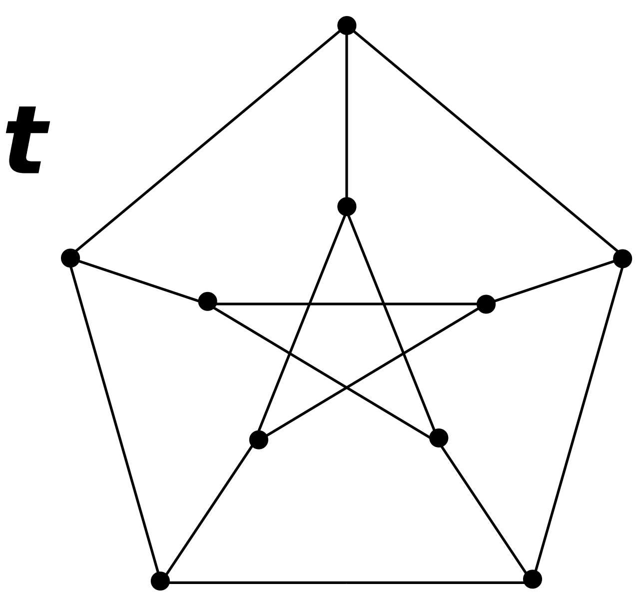


planar



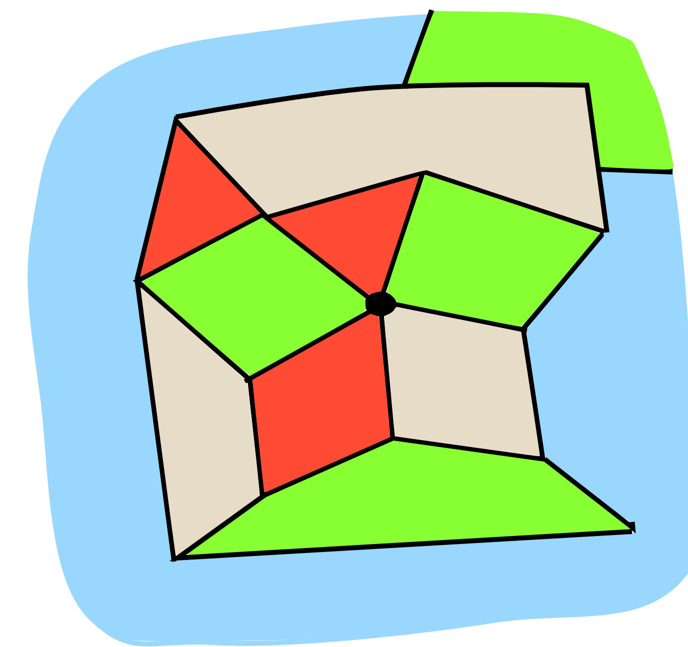
bridgeless

3-valent



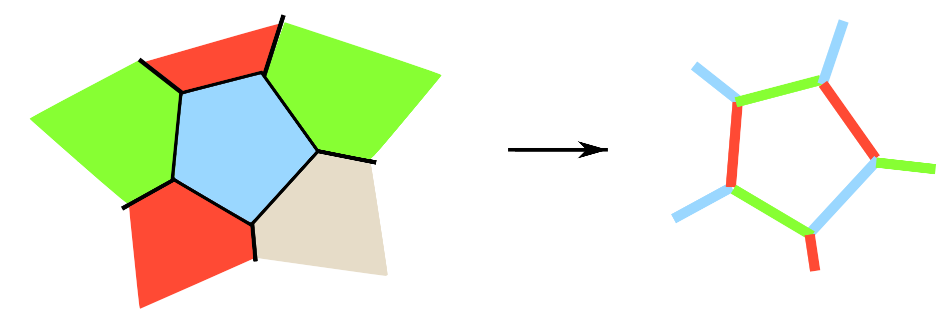
Four Colour Theorem

The 4CT is a statement about maps.

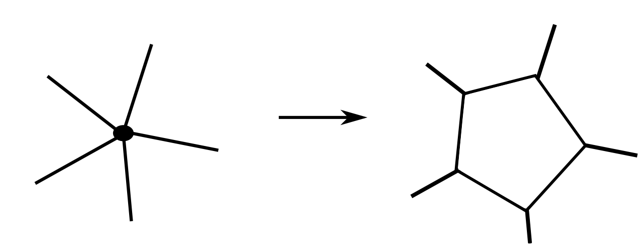


every bridgeless planar map
has a proper face 4-coloring

By a well-known reduction (Tait 1880), 4CT is equivalent to a statement about 3-valent maps



every bridgeless planar 3-valent map
has a proper edge 3-coloring



Map enumeration

From time to time in a graph-theoretical career one's thoughts turn to the Four Colour Problem. It occurred to me once that it might be possible to get results of interest in the theory of map-colourings without actually solving the Problem. For example, it might be possible to find the average number of colourings on vertices, for planar triangulations of a given size.

One would determine the number of triangulations of 2n faces, and then the number of 4-coloured triangulations of 2n faces. Then one would divide the second number by the first to get the required average. I gathered that this sort of retreat from a difficult problem to a related average was not unknown in other branches of Mathematics, and that it was particularly common in Number Theory.

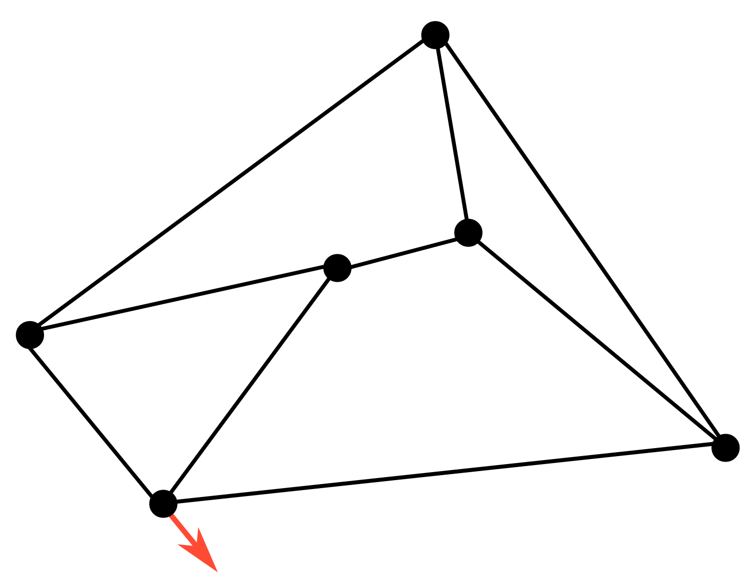
W. T. Tutte, Graph Theory as I Have Known It

Map enumeration

Tutte wrote a pioneering series of papers (1962-1969)

W. T. Tutte (1962), A census of planar triangulations. Canadian Journal of Mathematics 14:21–38
W. T. Tutte (1962), A census of Hamiltonian polygons. Can. J. Math. 14:402–417
W. T. Tutte (1962), A census of slicings. Can. J. Math. 14:708–722
W. T. Tutte (1963), A census of planar maps. Can. J. Math. 15:249–271
W. T. Tutte (1968), On the enumeration of planar maps. Bulletin of the American Mathematical Society 74:64–74
W. T. Tutte (1969), On the enumeration of four-colored maps. SIAM Journal on Applied Mathematics 17:454–460

One of his insights was to consider ***rooted*** maps



Key property: rooted maps have no non-trivial automorphisms

Map enumeration

Ultimately, Tutte obtained some remarkably simple formulas for counting different families of rooted planar maps.

(5.1) *The number a_n of rooted maps with n edges is*

$$\frac{2(2n)!\,3^n}{n!\,(n+2)!}.$$

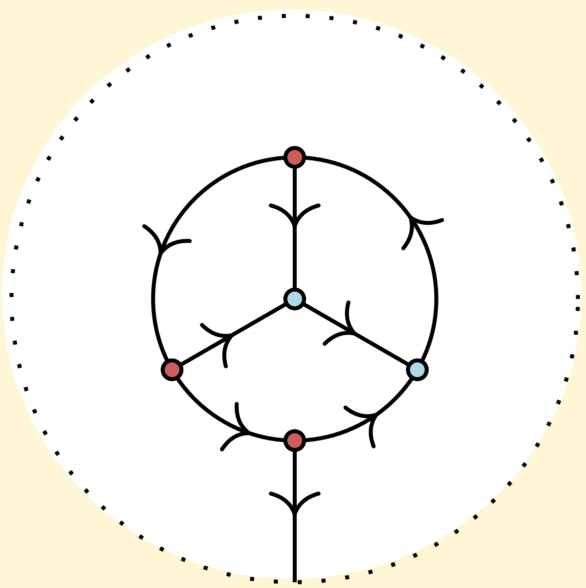
We write

$$A(x) = \sum_{n=1}^{\infty} a_n x^n.$$

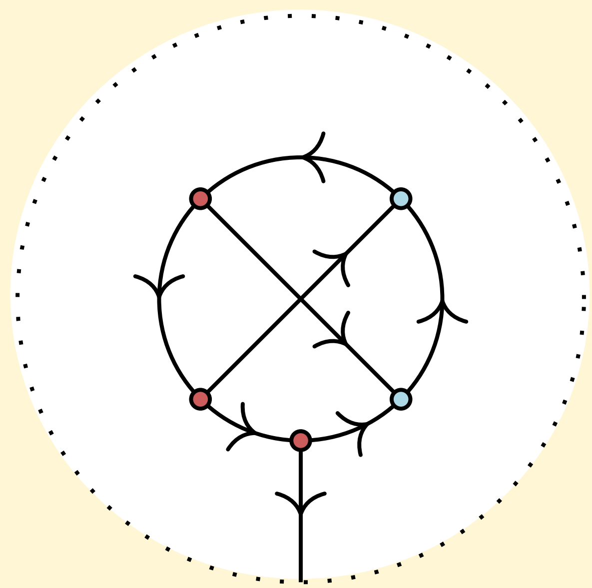
Thus $A(x) = 2x + 9x^2 + 54x^3 + 378x^4 + \ldots$. Figure 2 shows the 2 rooted maps with 1 edge, and Figure 3 the 9 rooted maps with 2 edges.

[Background]

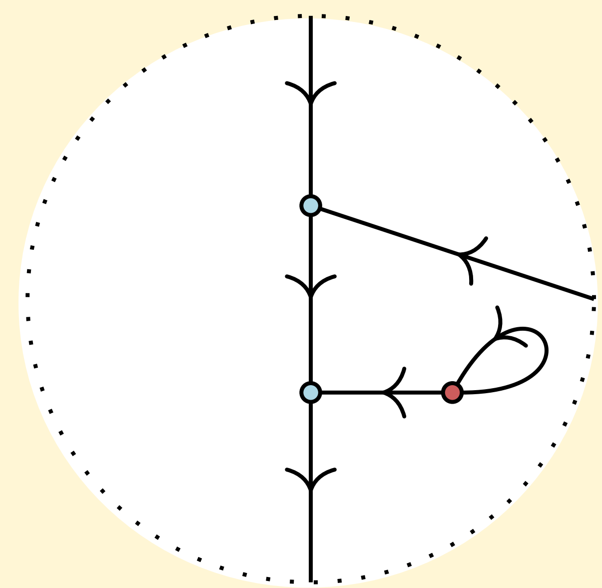
A few views on linear & planar λ-calculus



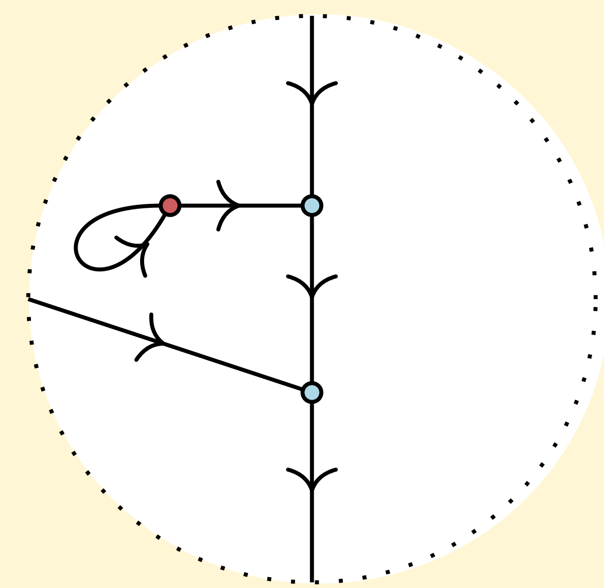
λx.λy.λz.x(yz)



λx.λy.λz.(xz)y



x,y ⊢ (xy)(λz.z)



x,y ⊢ x((λz.z)y)

Untyped lambda calculus in modern dress

pure lambda terms may be naturally organized into a *cartesian operad*

(cf. Hyland, "Classical lambda calculus in modern dress")

linear terms may be naturally organized into an ordinary (symmetric) operad

- $\Lambda(n)$ = set of α-equivalence classes of linear terms in context $x_1,\ldots,x_n \vdash t$

$$\frac{}{x \vdash x} \qquad \frac{\Gamma \vdash t \quad \Delta \vdash u}{\Gamma, \Delta \vdash t\ u} \qquad \frac{\Gamma, x \vdash t}{\Gamma \vdash \lambda x.t}$$

- $\circ_i : \Lambda(m+1) \times \Lambda(n) \to \Lambda(m+n)$ defined by (linear) substitution

$$\frac{\Theta \vdash u \qquad \Gamma,x,\Delta \vdash t}{\Gamma,\Theta,\Delta \vdash t[u/x]}$$

- symmetric action $S_n \times \Lambda(n) \to \Lambda(n)$ defined by permuting the context

$$\frac{\Gamma,y,x,\Delta \vdash t}{\Gamma,x,y,\Delta \vdash t}$$

Ordered & unitless terms

The operad of linear terms also has some natural *suboperads:*

- the *non-symmetric operad* of **ordered** ("planar") terms

$\lambda x.\lambda y.\lambda z.x(yz)$ but not $\lambda x.\lambda y.\lambda z.(xz)y$

- the *non-unitary operad* of terms with no closed subterms (**unitless**/"bridgeless")

$x \vdash \lambda y.yx$ but not $x \vdash x(\lambda y.y)$

(Can also combine these two restrictions.)

Linear typing

(NB: multicategory = colored operad)

typed linear terms may be interpreted as morphisms of a **closed multicategory**

$$\frac{}{x : A \vdash x : A} \qquad \frac{\Gamma \vdash t : A \multimap B \qquad \Delta \vdash u : A}{\Gamma, \Delta \vdash t\,u : B} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B}$$

(technically, to get a closed multicategory we need to quotient by βη)

the typed and untyped views are closely related...
1. every linear term can be typed
2. Λ is isomorphic to the endomorphism operad of a *reflexive object*

reflexive object in a closed (2-)category

Idea (after D. Scott): a linear lambda term may be interpreted as an endomorphism of a **reflexive object** in a (symmetric) closed category.

By a "reflexive object", we mean an object U equipped with a pair of operations

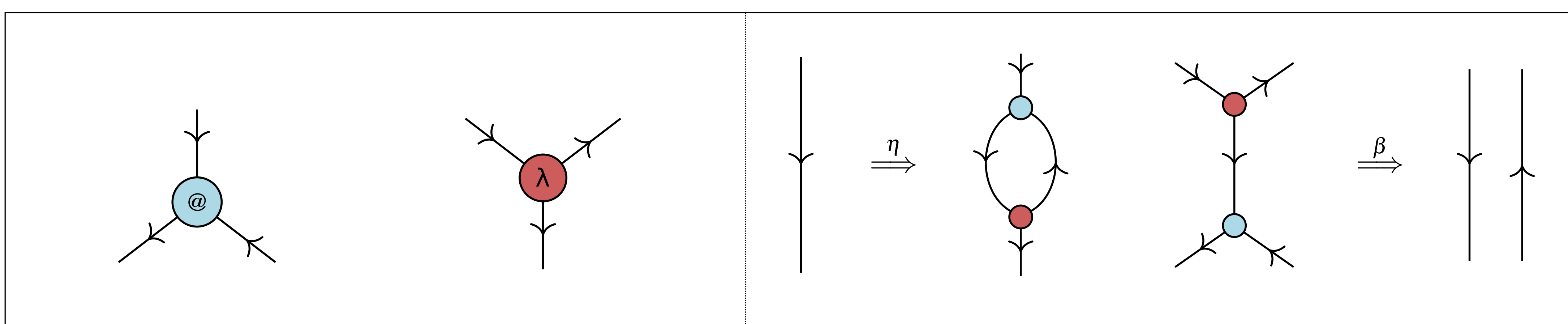
$$U \underset{\text{lam}}{\overset{\text{app}}{\rightleftarrows}} U \multimap U$$

which need not compose to the identity. Actually, it is natural to work in a closed 2-category and ask that these operations witness an *adjunction* from U to U ⊸ U. Then the unit and the counit of this adjunction respectively interpret ***η-expansion*** *t ⇒ λx.t(x)* and ***β-reduction*** *(λx.t)(u) ⇒ t[u/x].*

λ-graphs as string diagrams

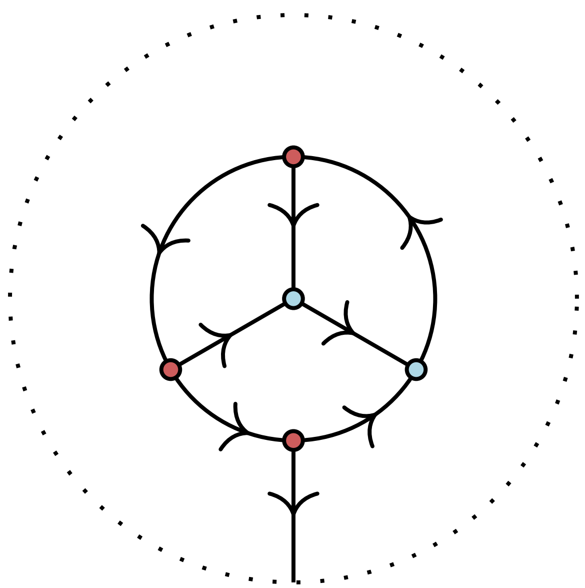
A *compact closed* (2-)category is a particular kind of closed (2-)category in which A ⊸ B ≈ B ⊗ A*. There are many natural examples, such as Rel, the (2-)category of sets and relations.

Compact closed categories have a well-known graphical language of "string diagrams". By expressing reflexive objects in this language, we recover the traditional diagram representing a linear term (cf. George's talk).

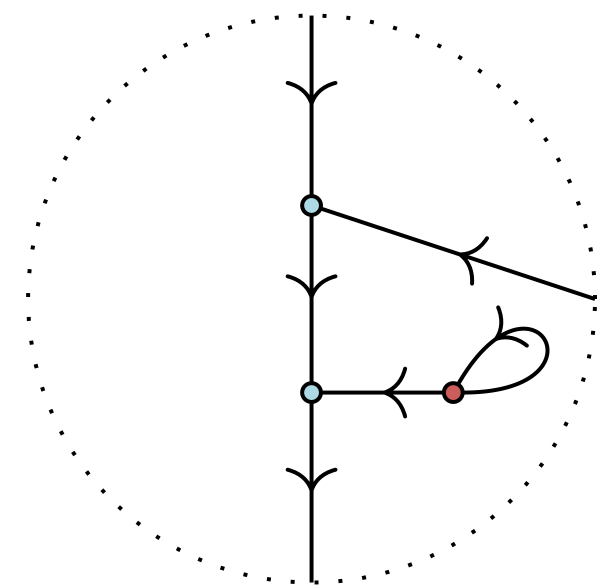


string diagrams as HOAS

Another way of putting this is that these diagrams are closely related to the representation of λ-terms using *higher-order abstract syntax*



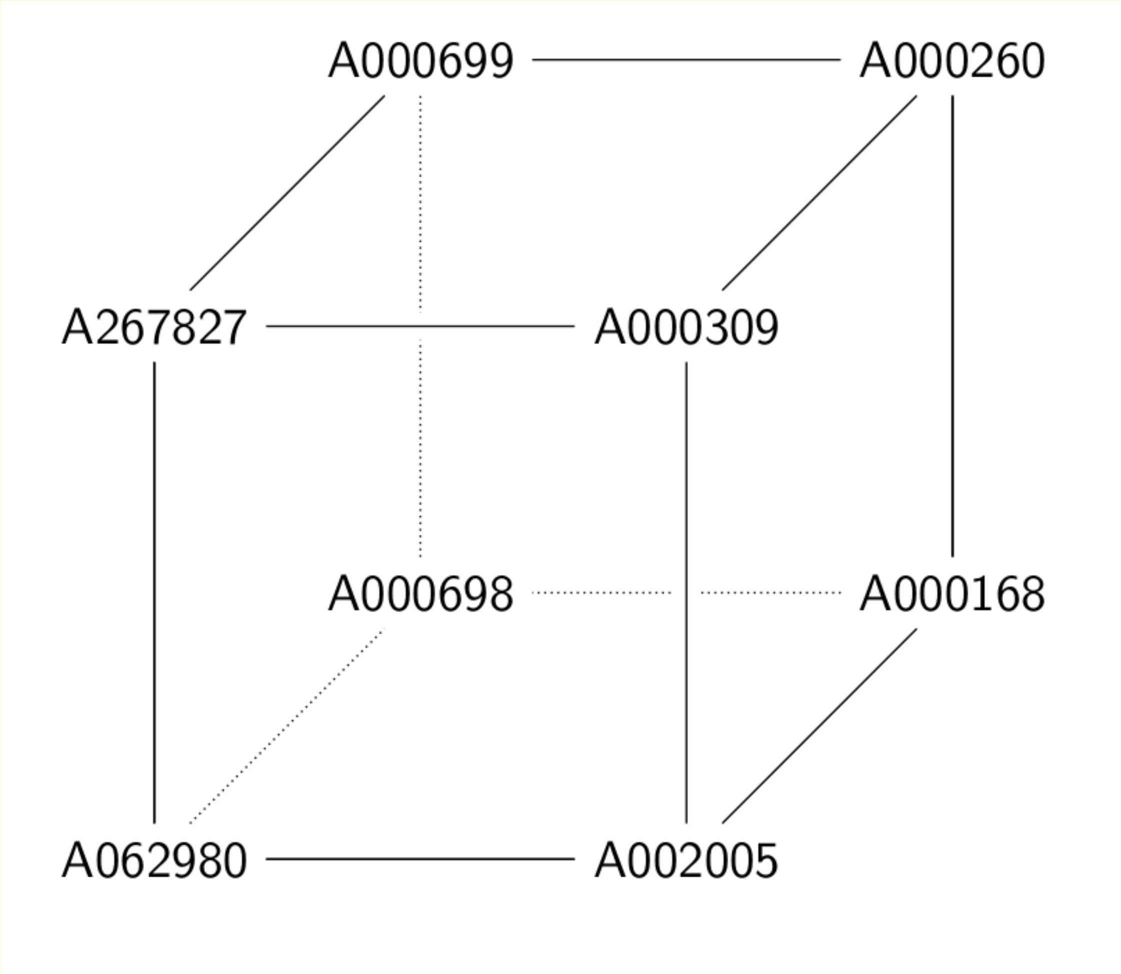
lam λx.lam λy.lam λz.app x (app y z)



λx.λy.app (app x y)(lam λz.z)

[Background]

Enumera- & bijective connections



family of rooted maps	family of lambda terms	sequence	OEIS
planar maps	normal ordered terms	1,2,9,54,378,2916,...	A000168

Z, A. Giorgetti (2015), A correspondence between rooted planar maps and normal planar lambda terms, LMCS 11(3:22): 1-39

Some enumerative connections

family of rooted maps	family of lambda terms	sequence	OEIS
trivalent maps (genus g≥0)	linear terms	1,5,60,1105,27120,...	A062980
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1. O. Bodini, D. Gardy, A. Jacquot (2013), Asymptotics and random sampling for BCI and BCK lambda terms, TCS 502: 227-238
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planar trivalent maps	ordered terms	1,4,32,336,4096,...	A002005
bridgeless trivalent maps	unitless linear terms	1,2,20,352,8624,...	A267827
bridgeless planar trivalent maps	unitless ordered terms	1,1,4,24,176,1456,...	A000309
maps (genus g≥0)	normal linear terms (mod ~)	1,2,10,74,706,8162,...	A000698
planar maps	normal ordered terms	1,2,9,54,378,2916,...	A000168
bridgeless maps	normal unitless linear terms (mod ~)	1,1,4,27,248,2830,...	A000699
bridgeless planar maps	normal unitless ordered terms	1,1,3,13,68,399,...	A000260

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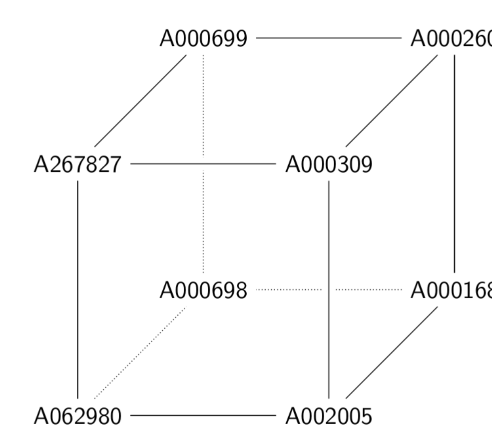
3. Z (2015), Counting isomorphism classes of beta-normal linear lambda terms, arXiv:1509.07596
4. Z (2016), Linear lambda terms as invariants of rooted trivalent maps, J. Functional Programming 26(e21)
5. J. Courtiel, K. Yeats, Z (2016), Connected chord diagrams and bridgeless maps, arXiv:1611.04611
6. Z (2017), A sequent calculus for a semi-associative law, FSCD



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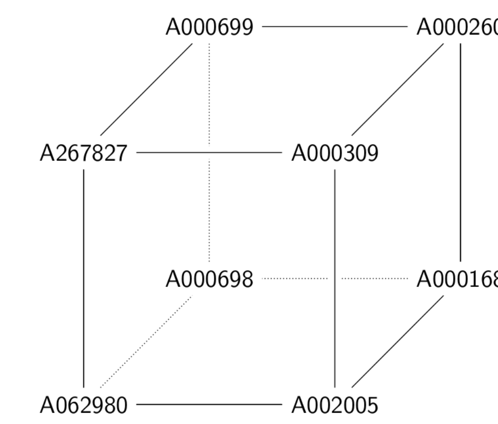
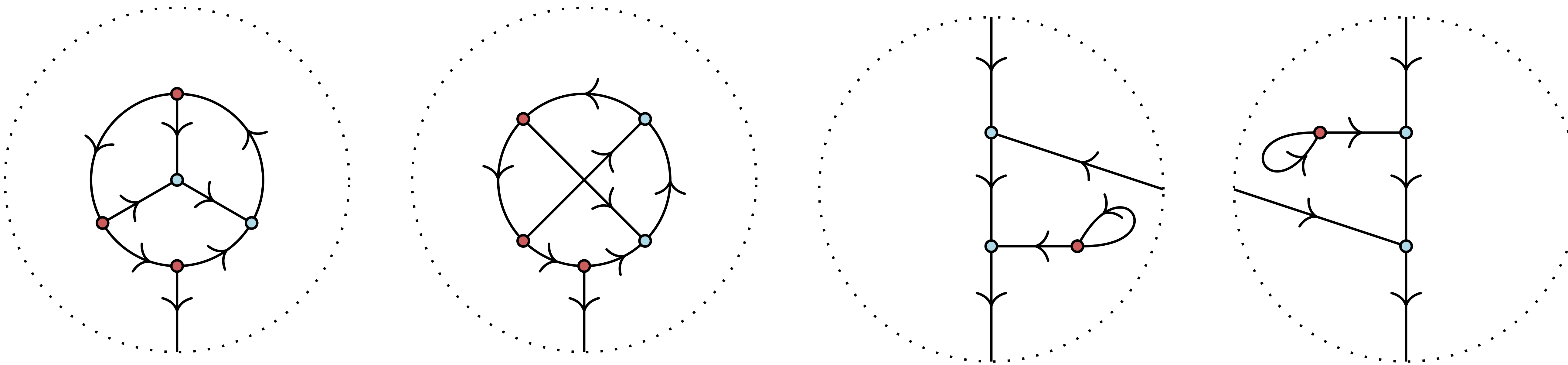


From linear terms to rooted 3-valent maps via string diagrams

$\lambda x.\lambda y.\lambda z.x(yz)$ $\lambda x.\lambda y.\lambda z.(xz)y$ $x,y \vdash (xy)(\lambda z.z)$ $x,y \vdash x((\lambda z.z)y)$

From linear terms to rooted 3-valent maps via string diagrams



$\lambda x.\lambda y.\lambda z.x(yz)$ $\quad$ $\lambda x.\lambda y.\lambda z.(xz)y$ $\quad$ $x,y \vdash (xy)(\lambda z.z)$ $\quad$ $x,y \vdash x((\lambda z.z)y)$

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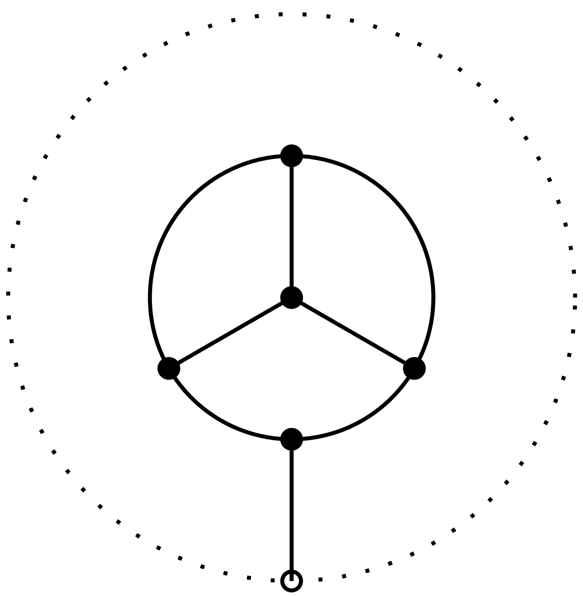
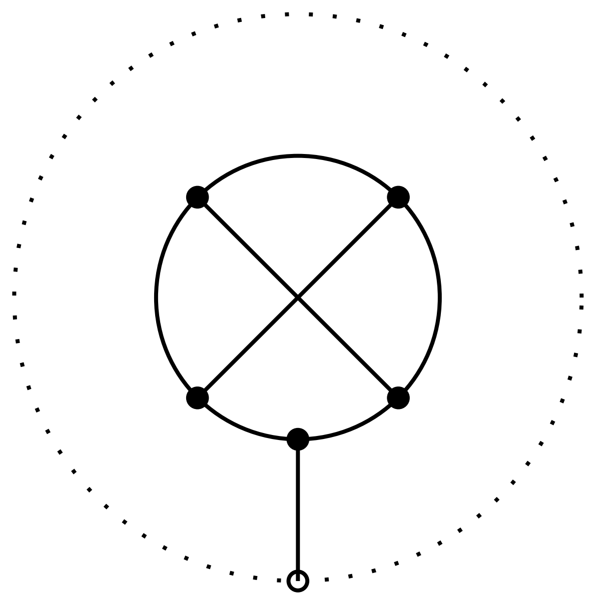
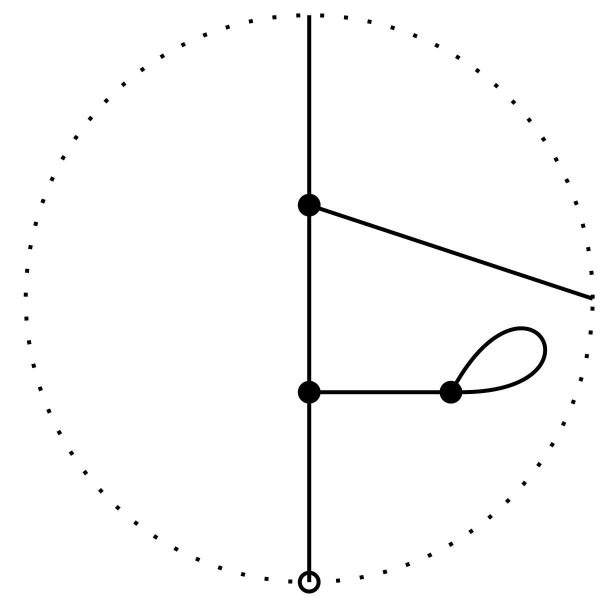


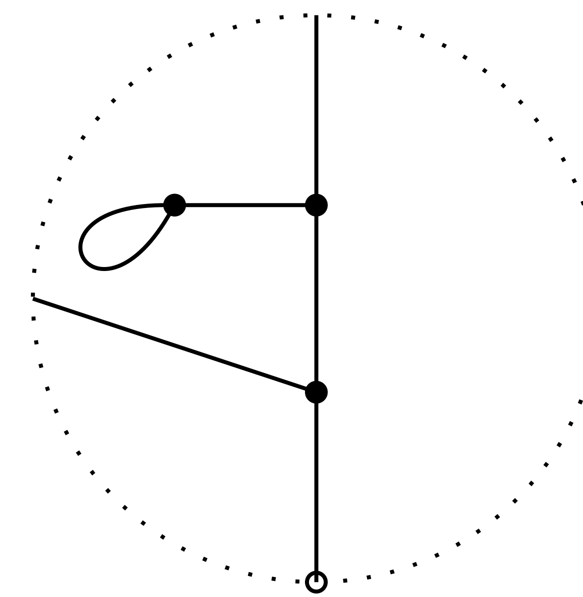
$\lambda x.\lambda y.\lambda z.x(yz)$



$\lambda x.\lambda y.\lambda z.(xz)y$



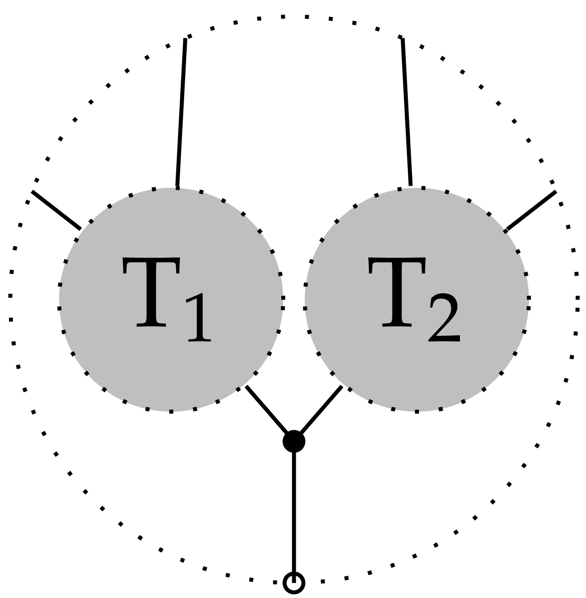
$x,y \vdash (xy)(\lambda z.z)$



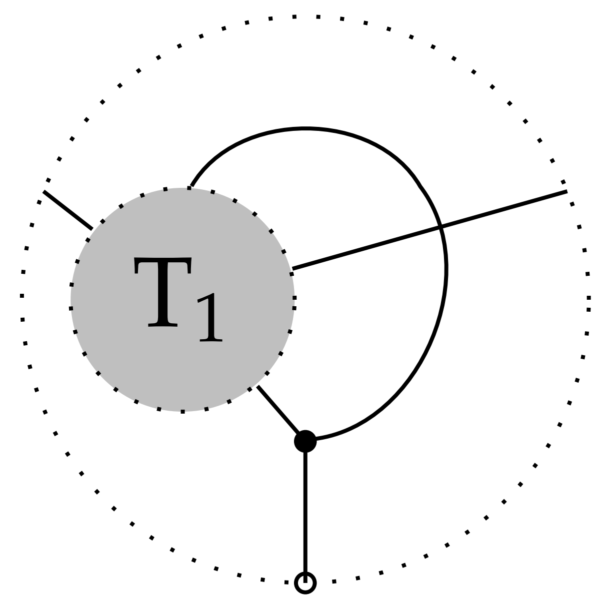
$x,y \vdash x((\lambda z.z)y)$

From rooted 3-valent maps to linear terms by induction

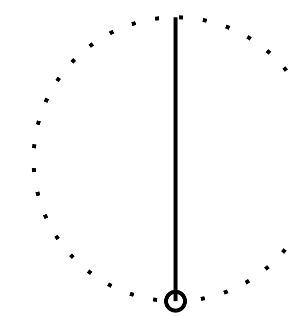
Observation: any rooted 3-valent map must have one of the following forms.



disconnecting
root vertex



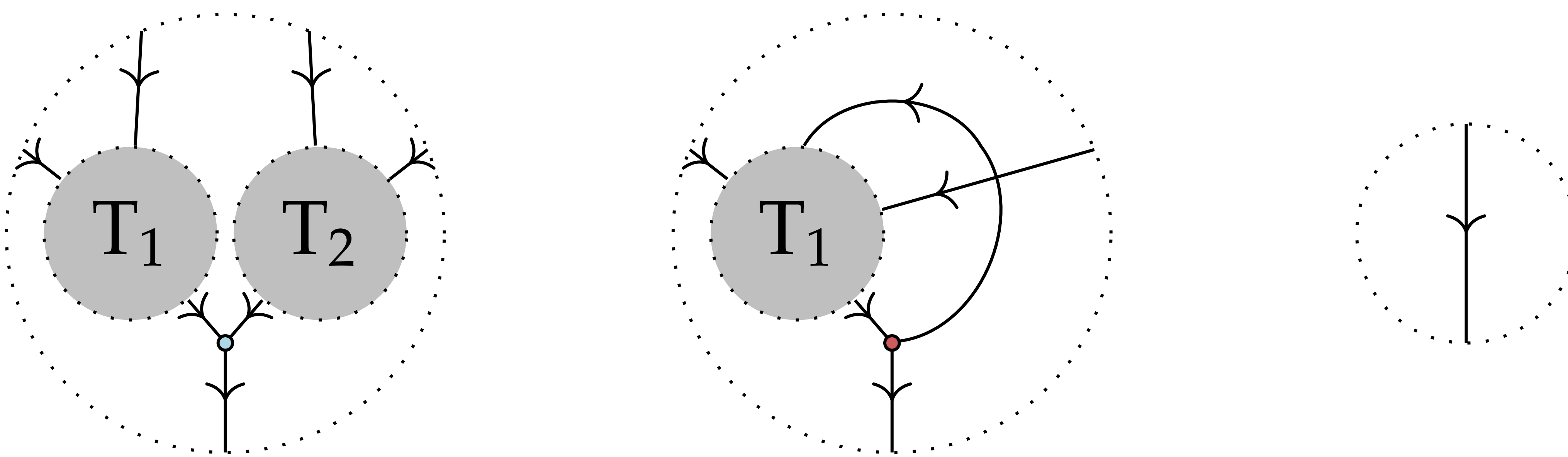
connecting
root vertex



no
root vertex

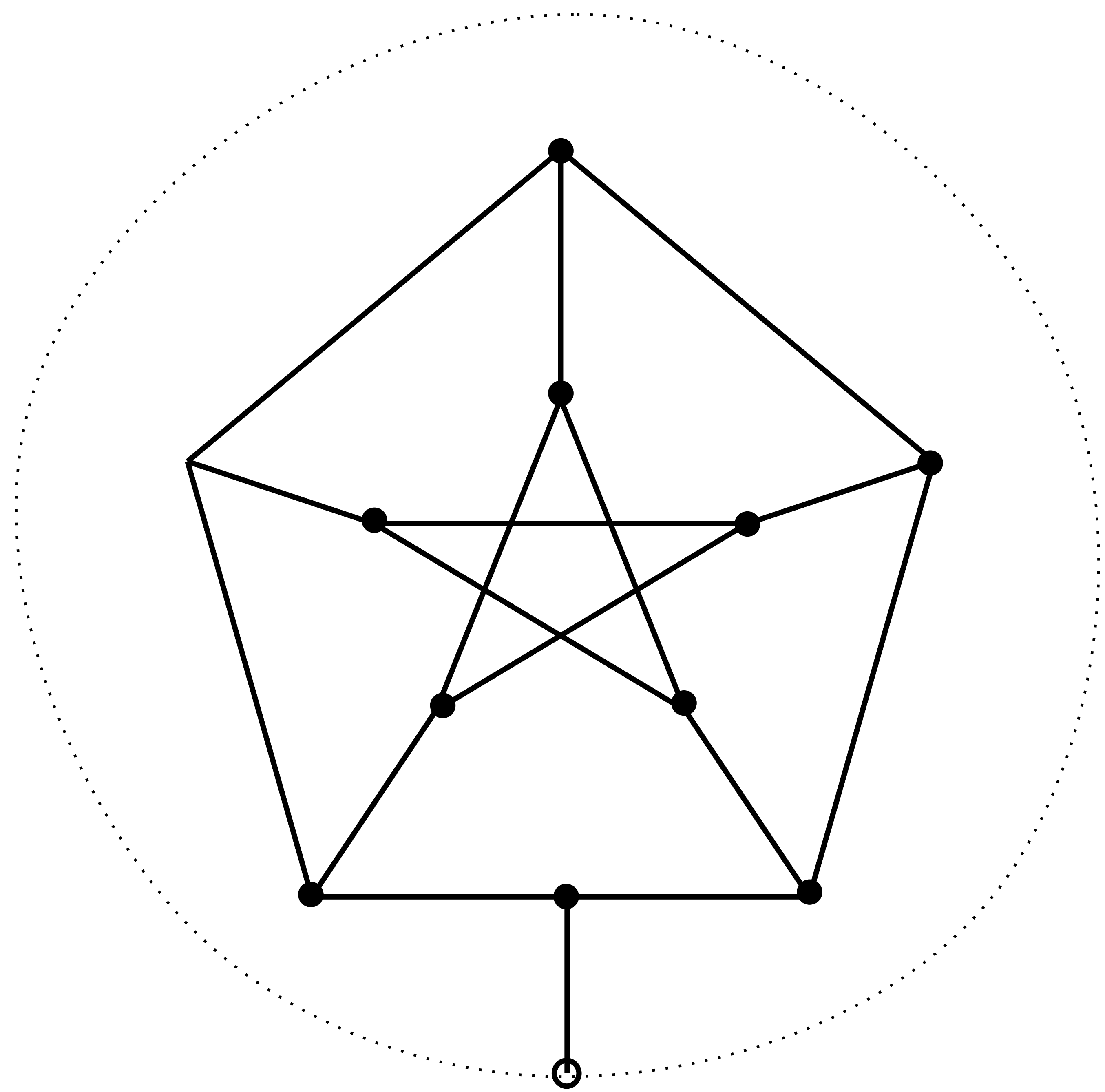
From rooted 3-valent maps to linear terms by induction

...but this exactly mirrors the inductive structure of linear lambda terms!



application abstraction variable

An example



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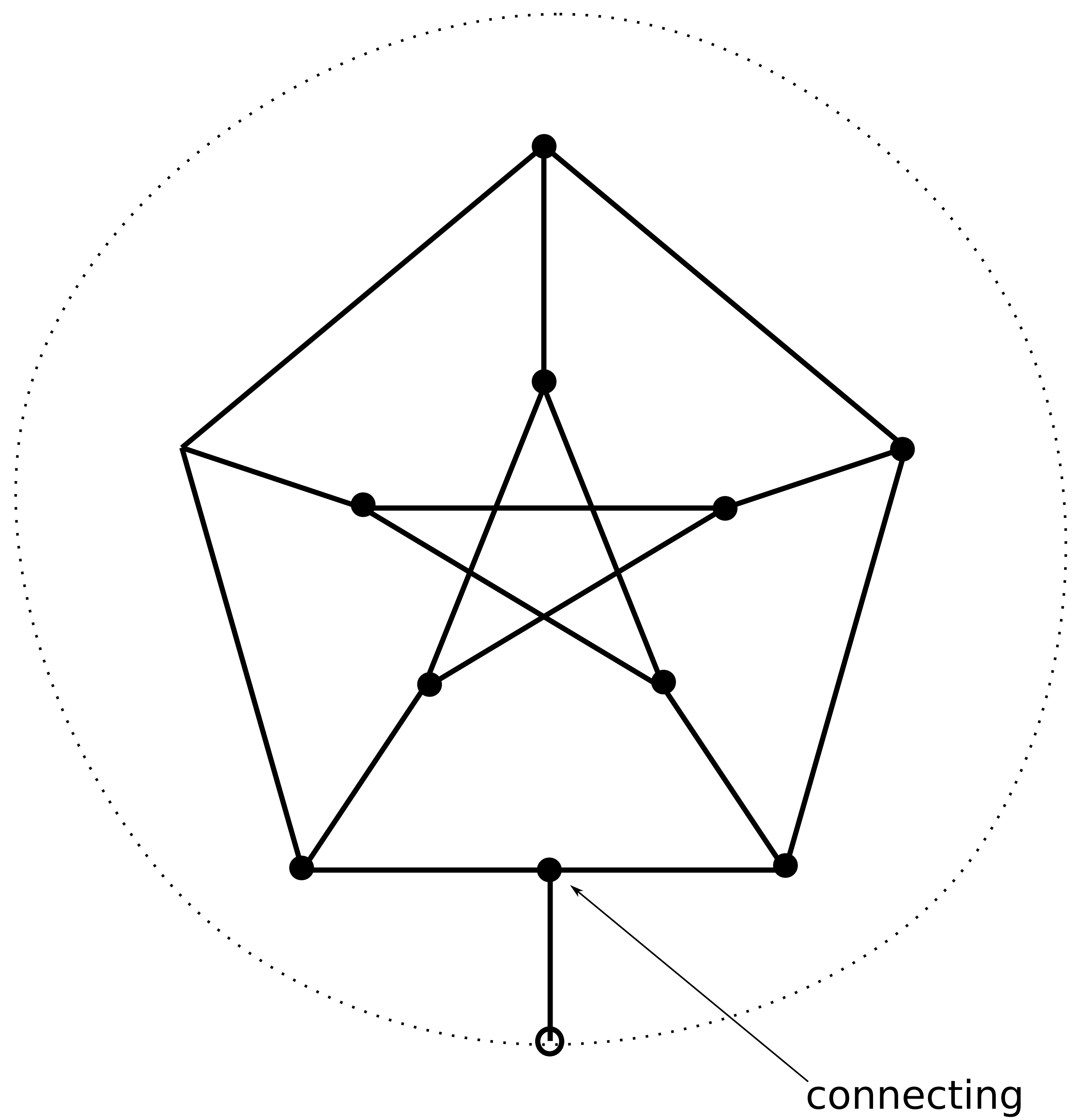

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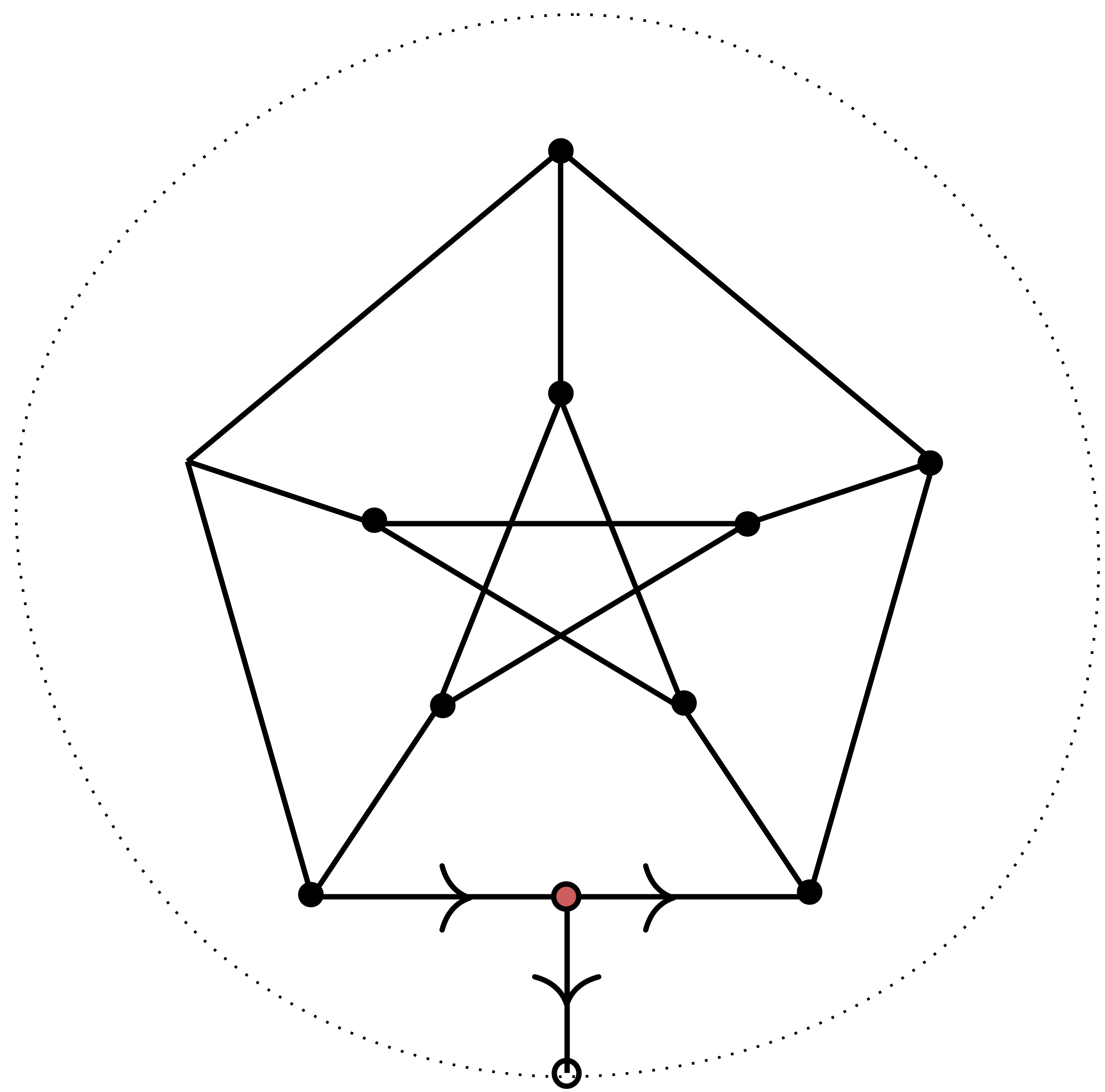

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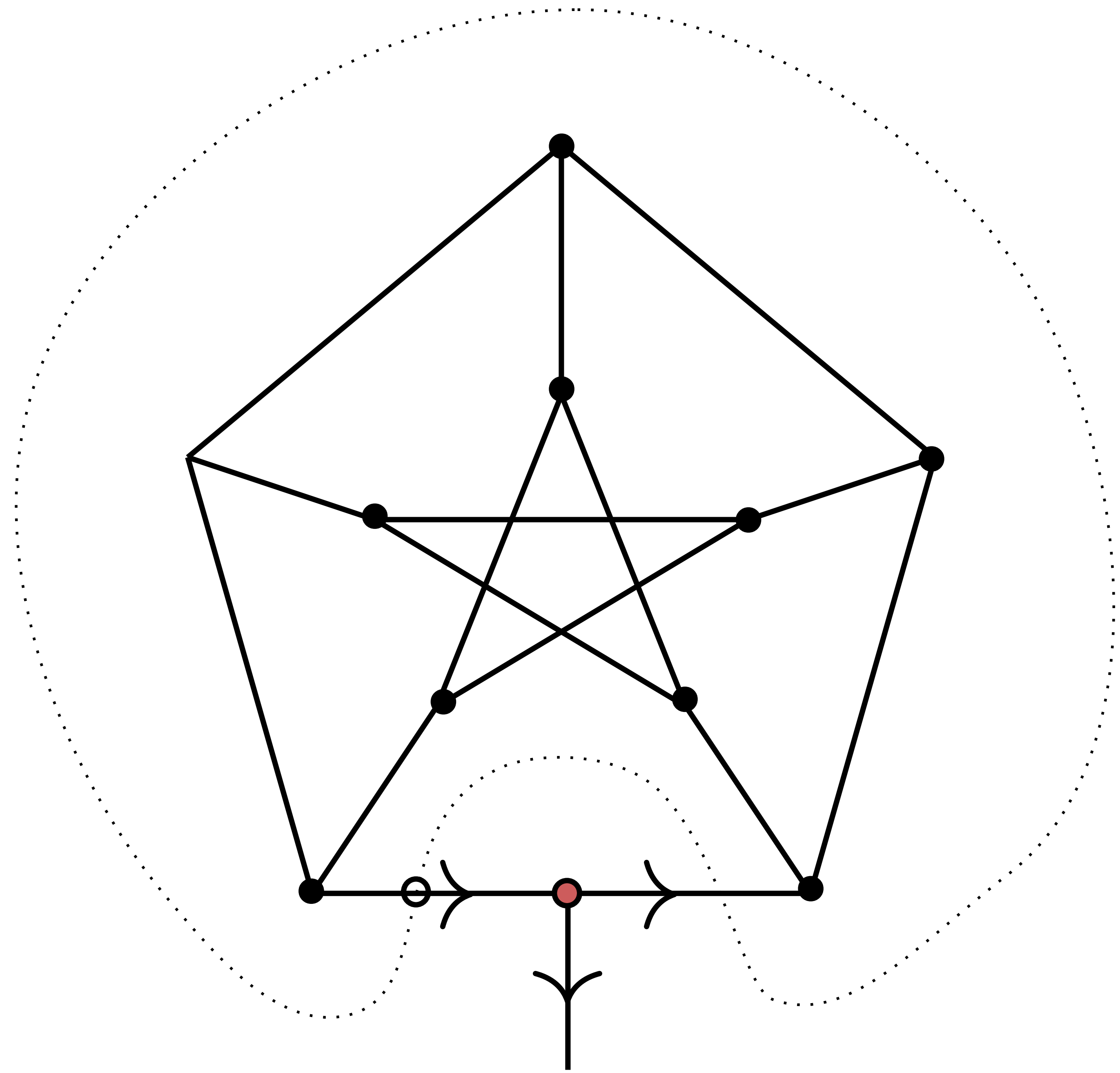

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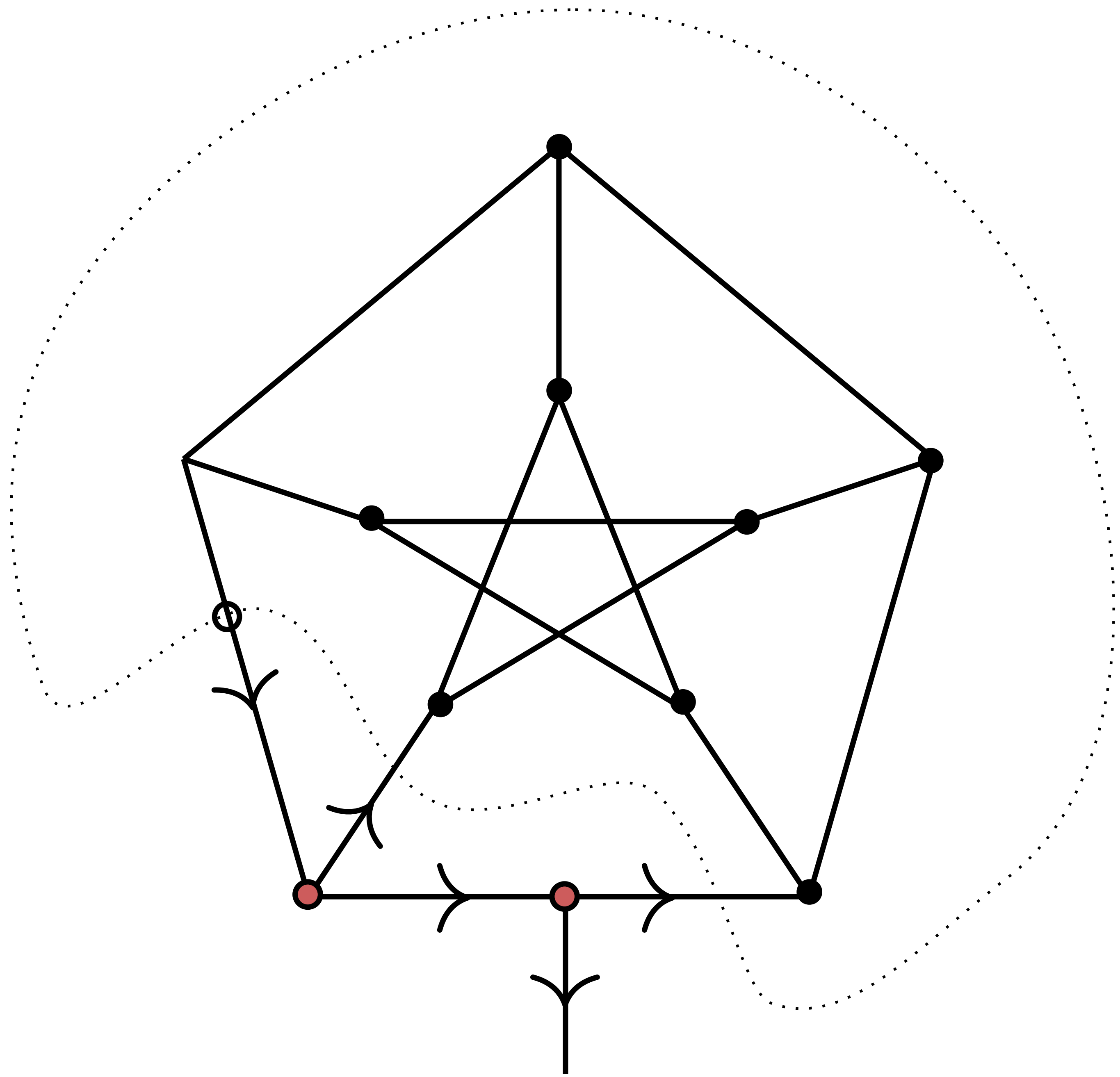

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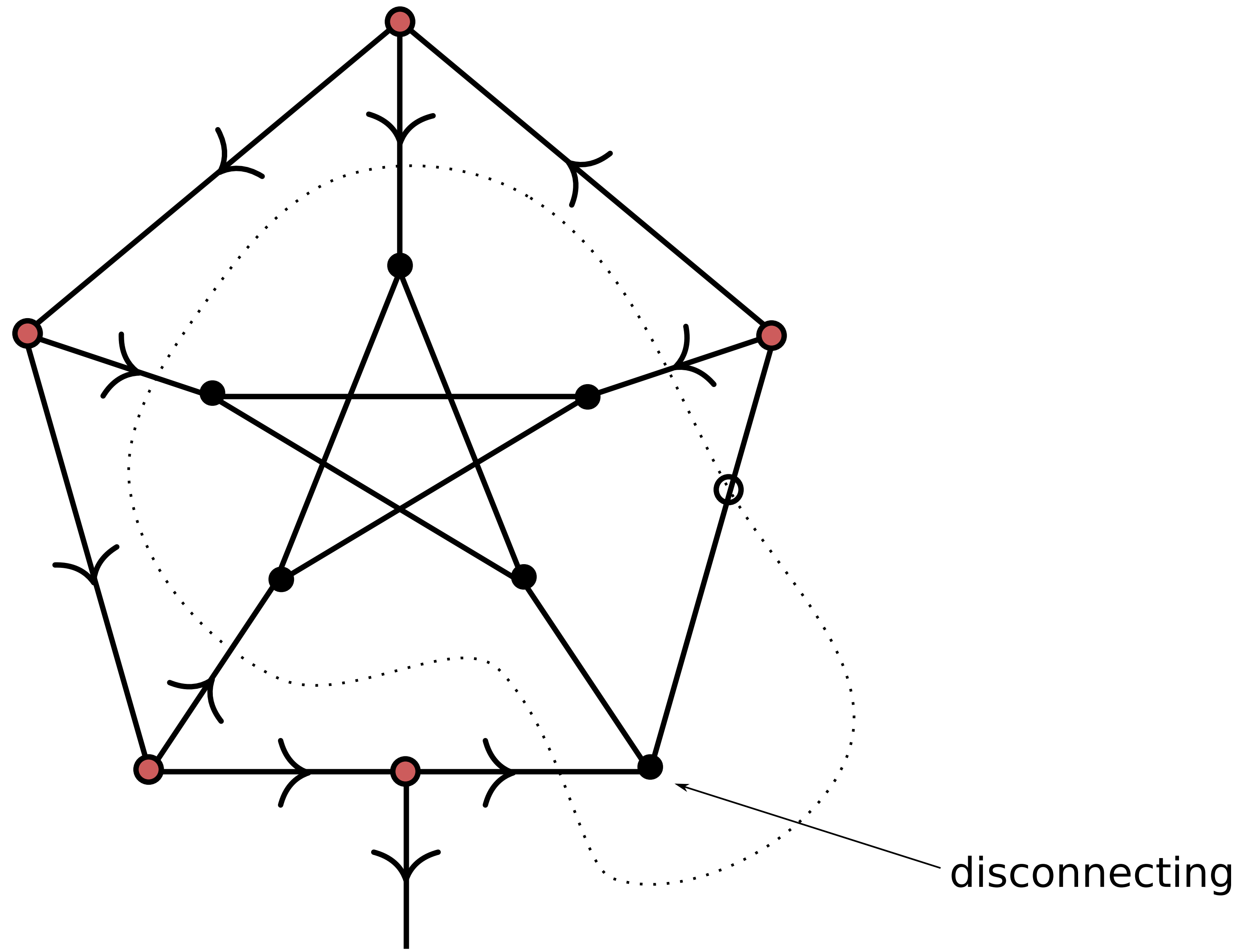

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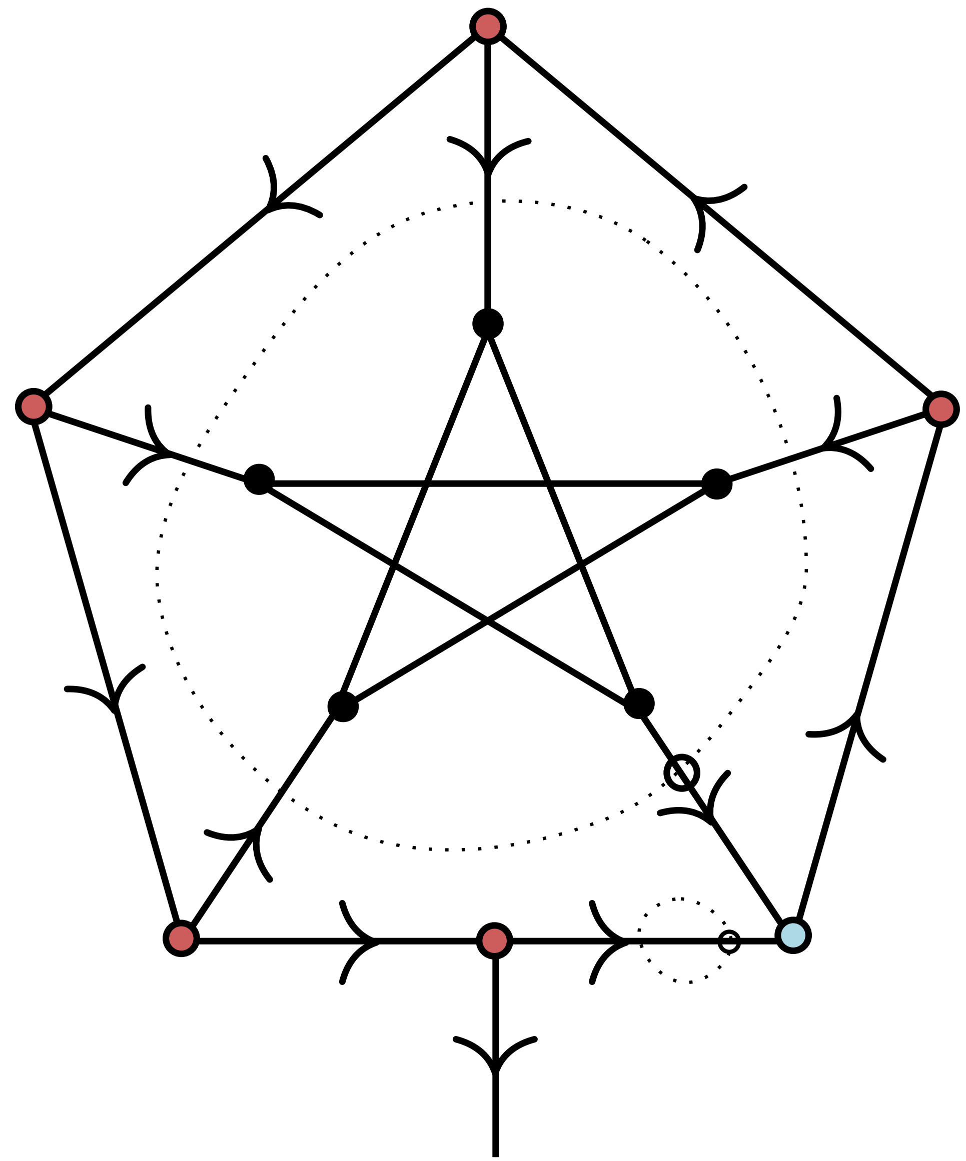
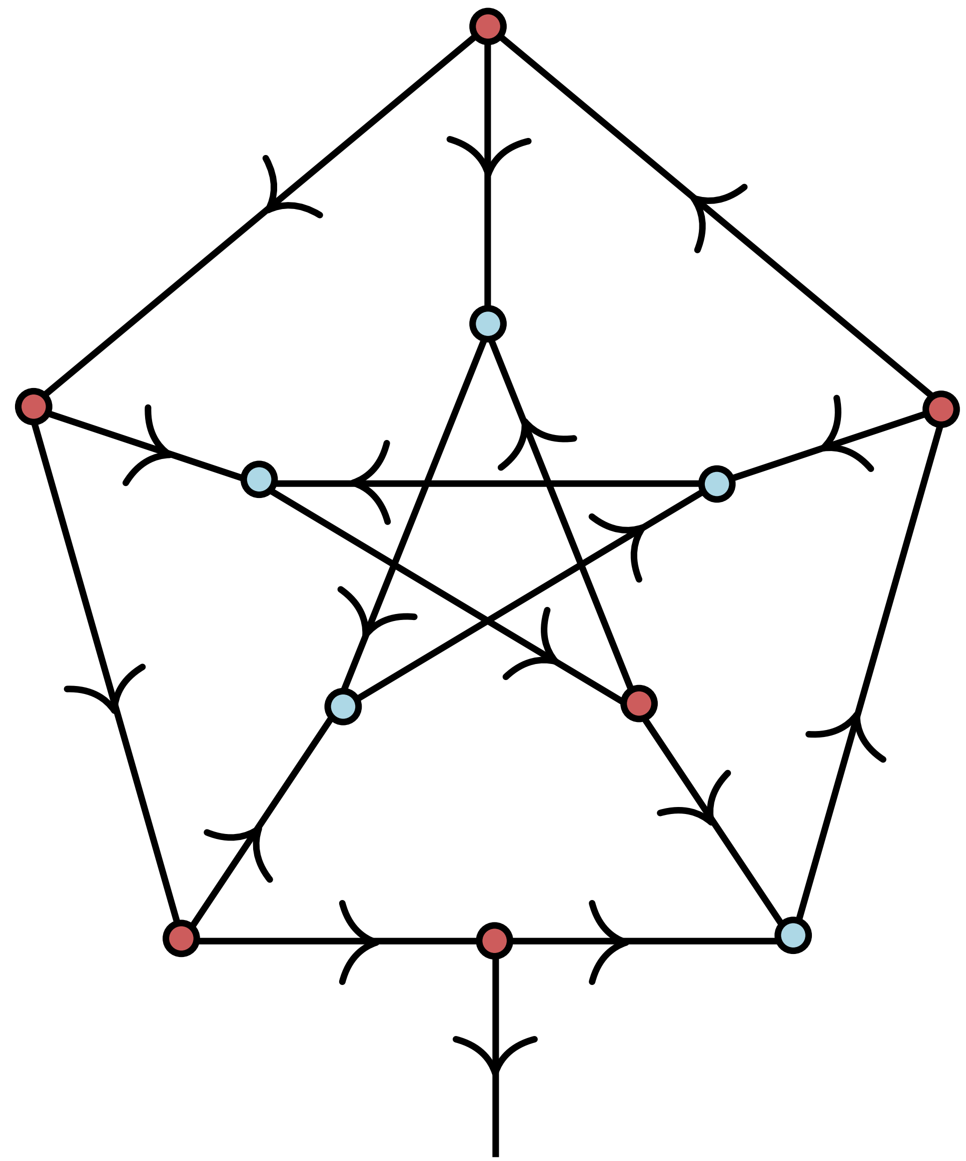
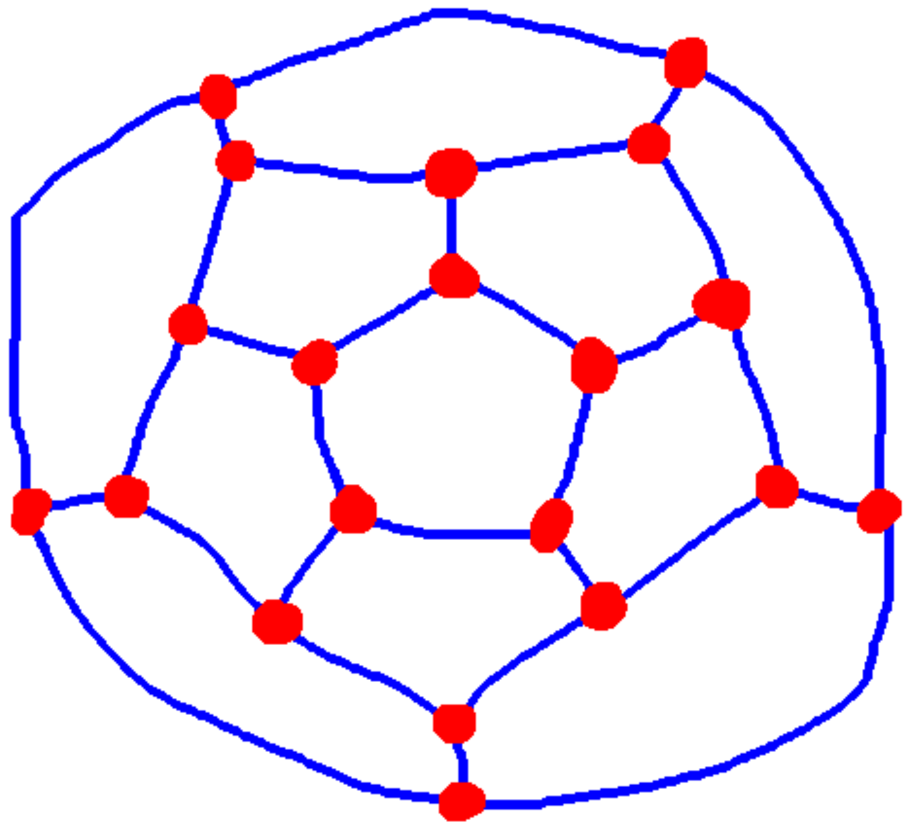


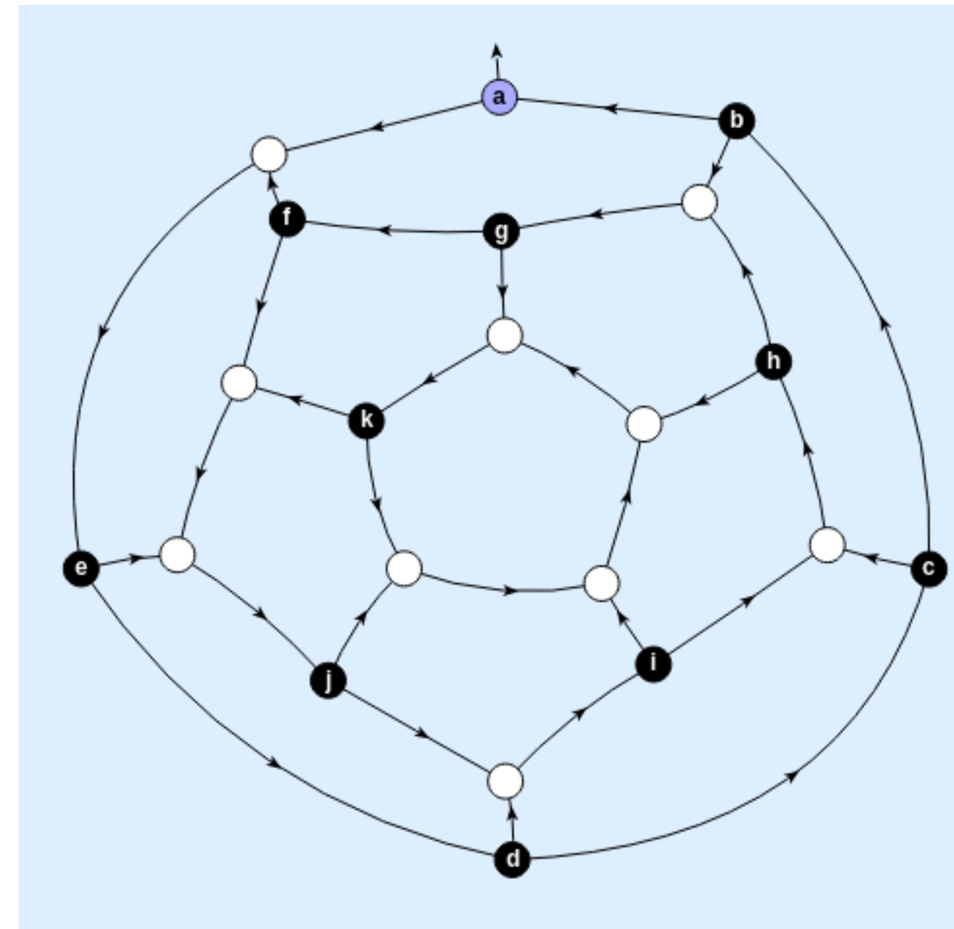
An example



λa.λb.λc.λd.λe.a(λf.c(e(b(df))))

Some more examples*

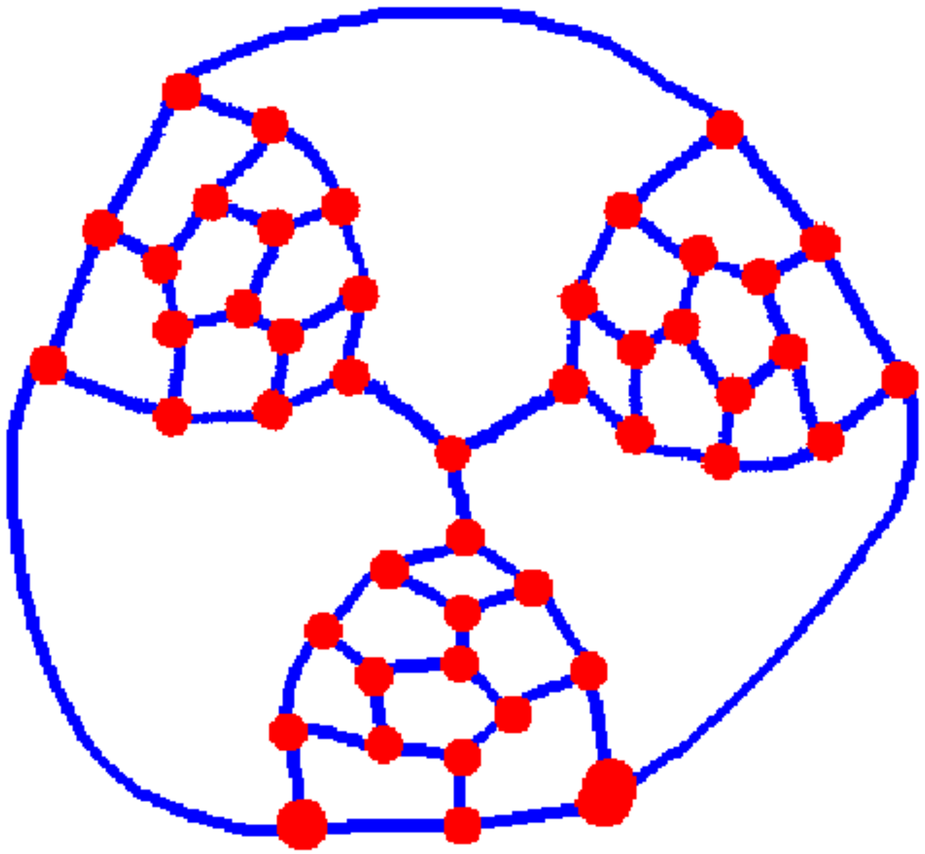


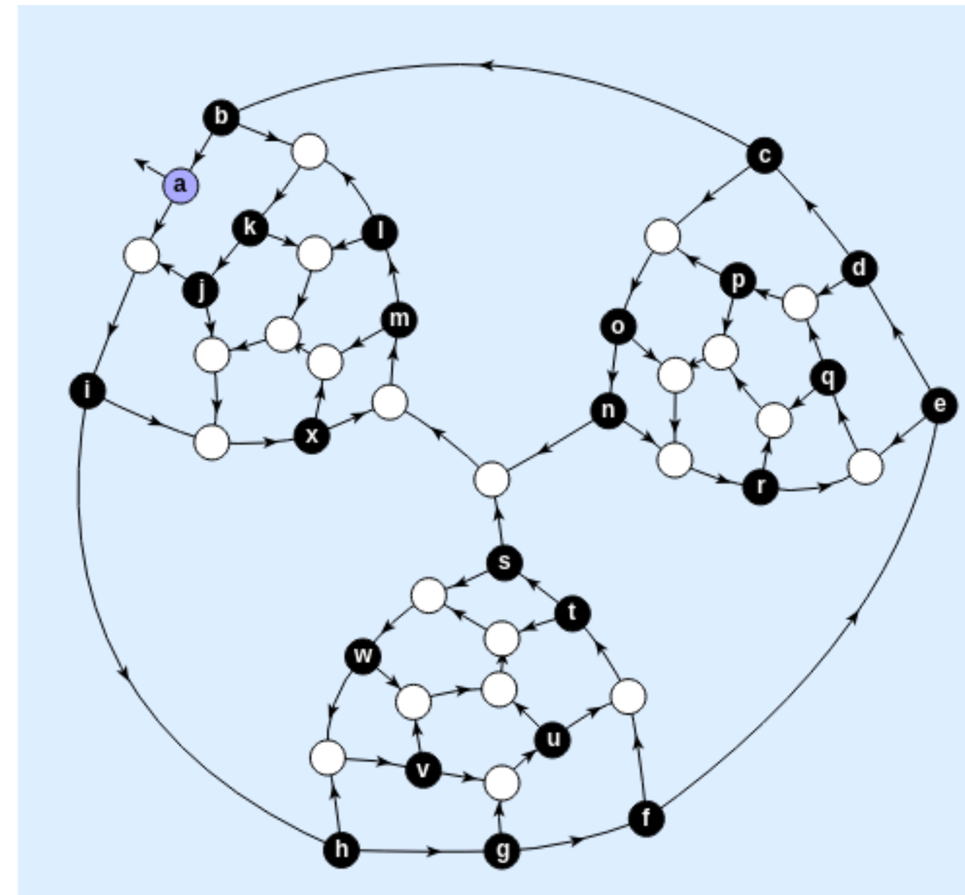


λabcde.a (λfg.b (λh.c (λi.d (λj.e (f (λk.g (h (i (j k))))))))

*computed with the help of https://jcreedcmu.github.io/demo/lambda-map-drawer/public/index.html

Some more examples*

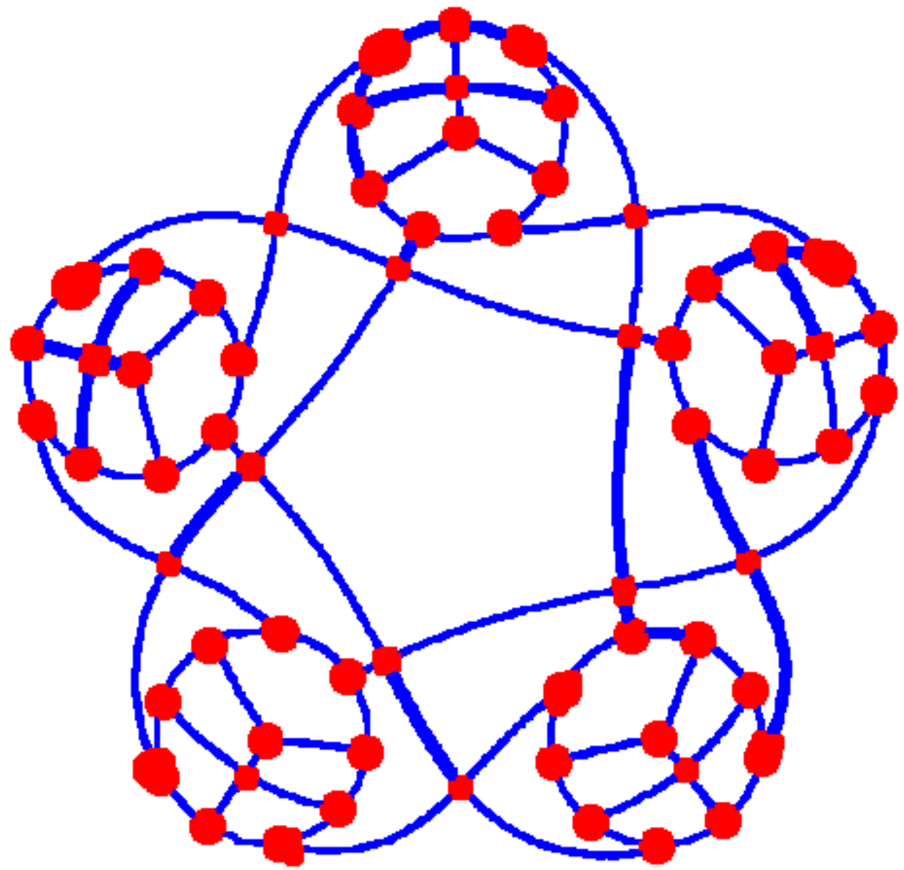
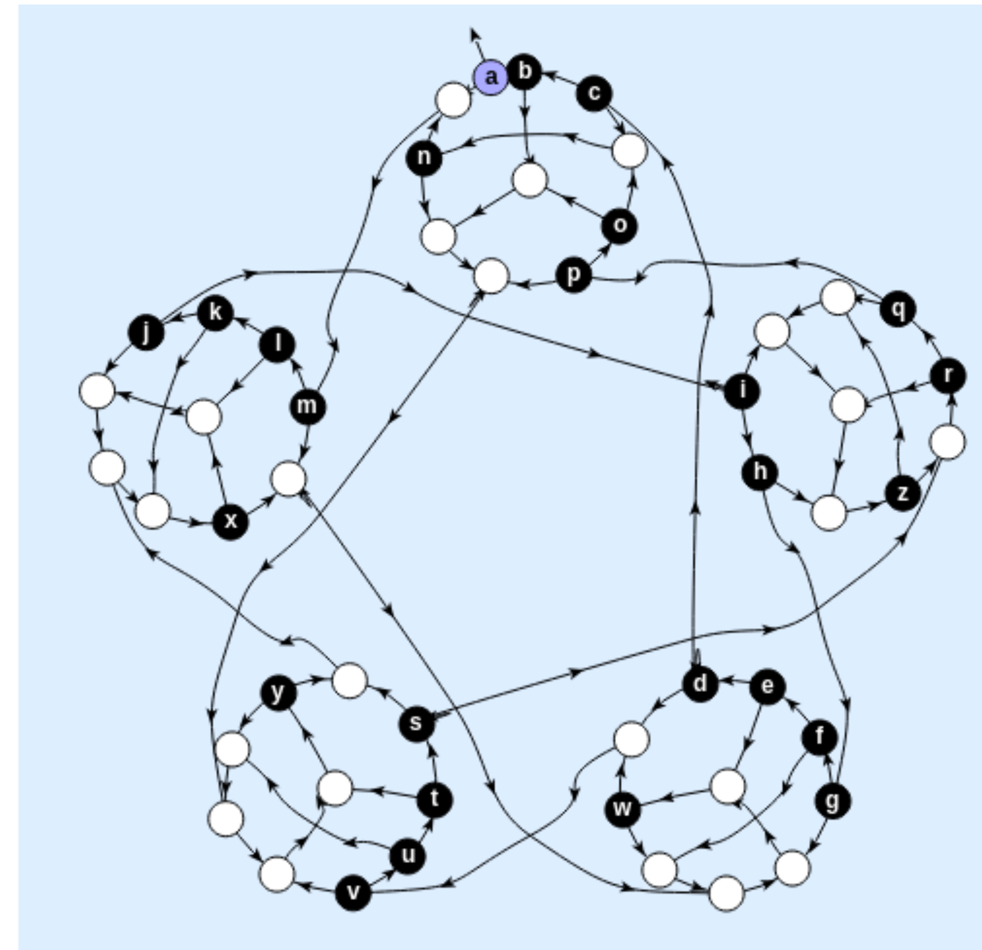




λabcdefghi.a (λjk.b (λlm.(λno.c (λp.d (λq.e (λr.n (o (p (q r))))))) (λst.f (λu.g (λv.h (λw.s (t (u (v w))))))) (λx.i (j (k l (m x))))))

*computed with the help of https://jcreedcmu.github.io/demo/lambda-map-drawer/public/index.html

Some more examples*

λabcdefghijklm.a (λn.c (λopqr.(λstuv.d (λw.e (g ((λx.s (λy.t (v (n (b o) p (y u)))) (j (l x)) k) m (w f))))) (λz.h (i (q z) r))))

*computed with the help of https://jcreedcmu.github.io/demo/lambda-map-drawer/public/index.html

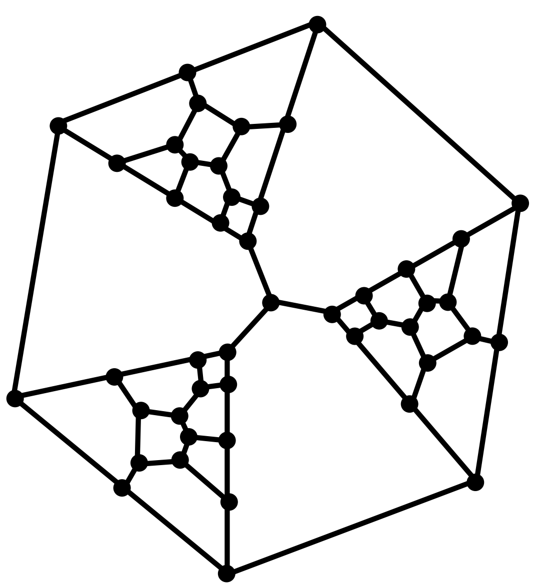
Some more analysis

the bijection 3-valent maps ↔ linear terms restricts to the suboperads

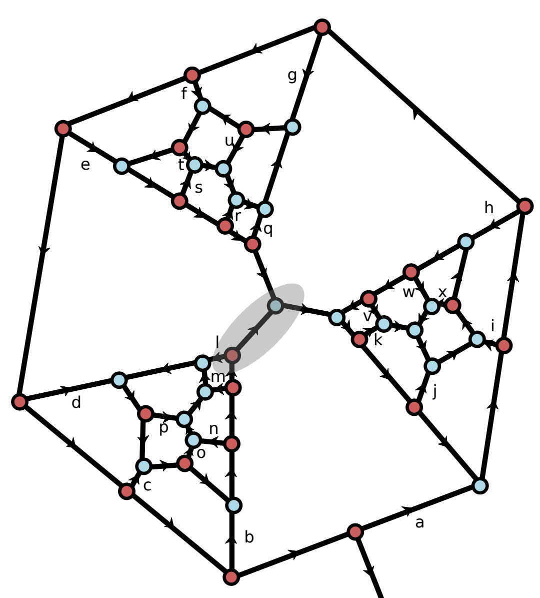
planar ↔ ordered   bridgeless ↔ unitless

typing corresponds to *edge-coloring* (cf. JFP 2016, LICS 2018)

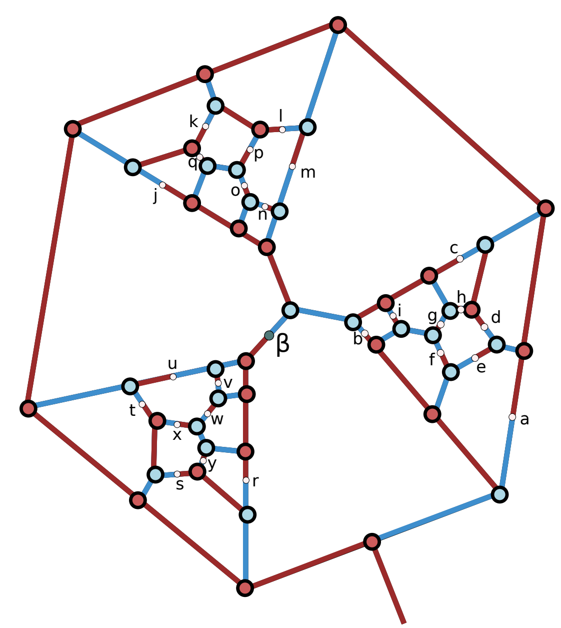
...indeed, there is a natural λ-formulation of 4CT!



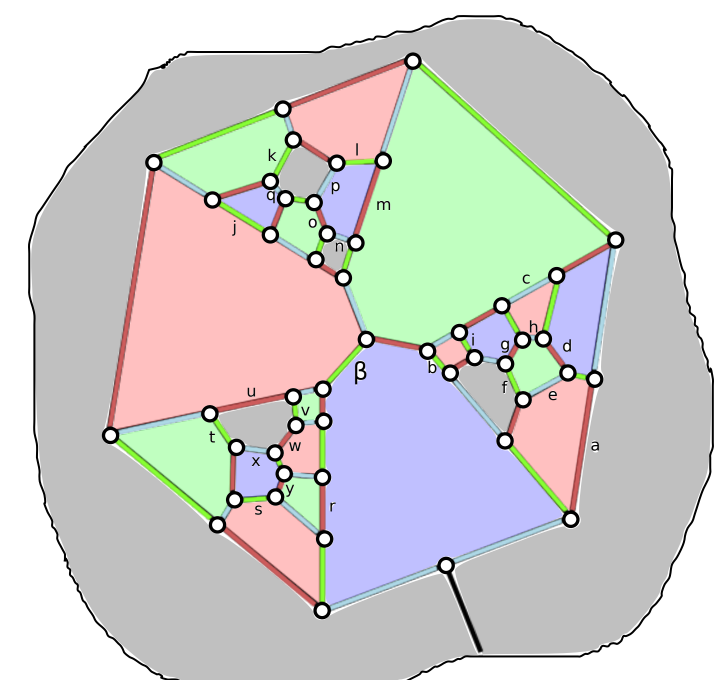
3-valent map



linear lambda term



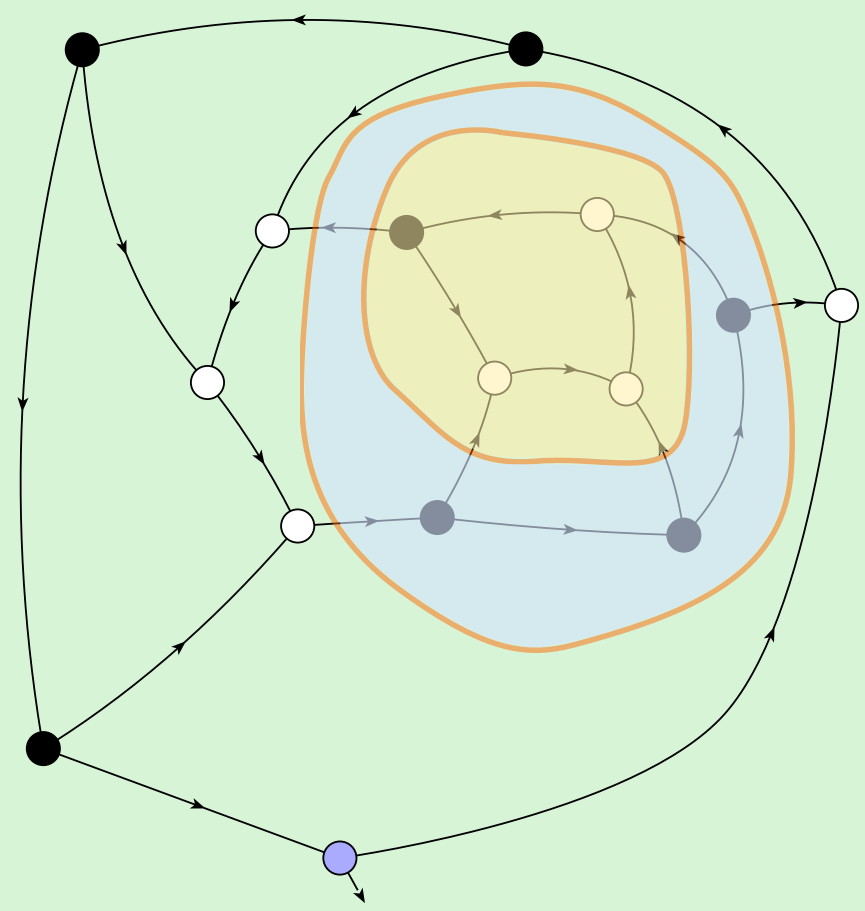
principal typing



𝕍-typing

[work-in-progress]

Connectivity in λ-calculus



k-edge-connection

a graph is **k-edge-connected** if it stays connected after cutting any $j < k$ edges

1-edge-connected = connected

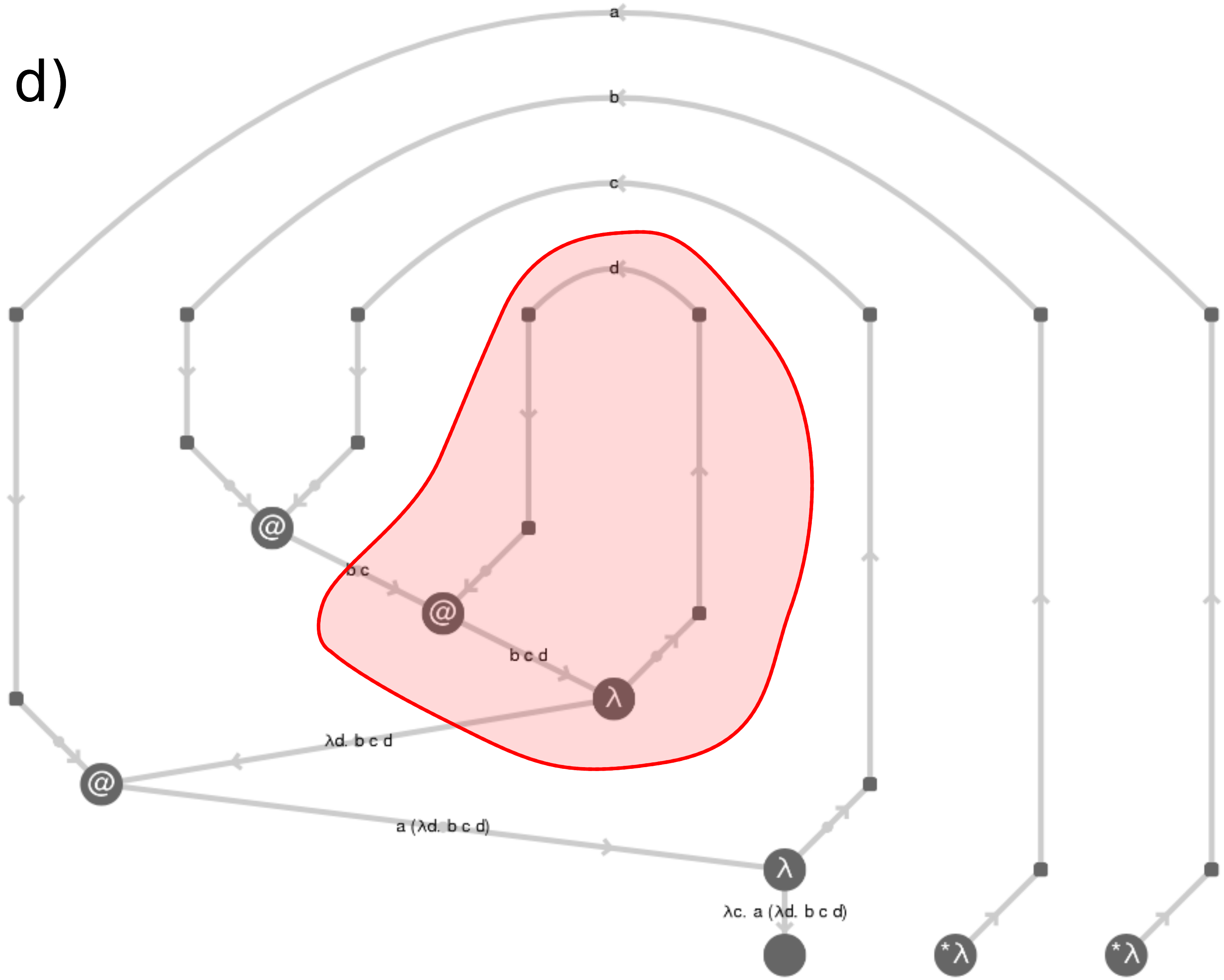
2-edge-connected = bridgeless

a 3-valent graph cannot be 4-edge-connected, but it can be **internally** 4-edge-connected (only trivial 3-cuts).

what does it mean for a linear λ-term to be internally k-edge-connected?

a term which is 2- but not 3-edge-connected

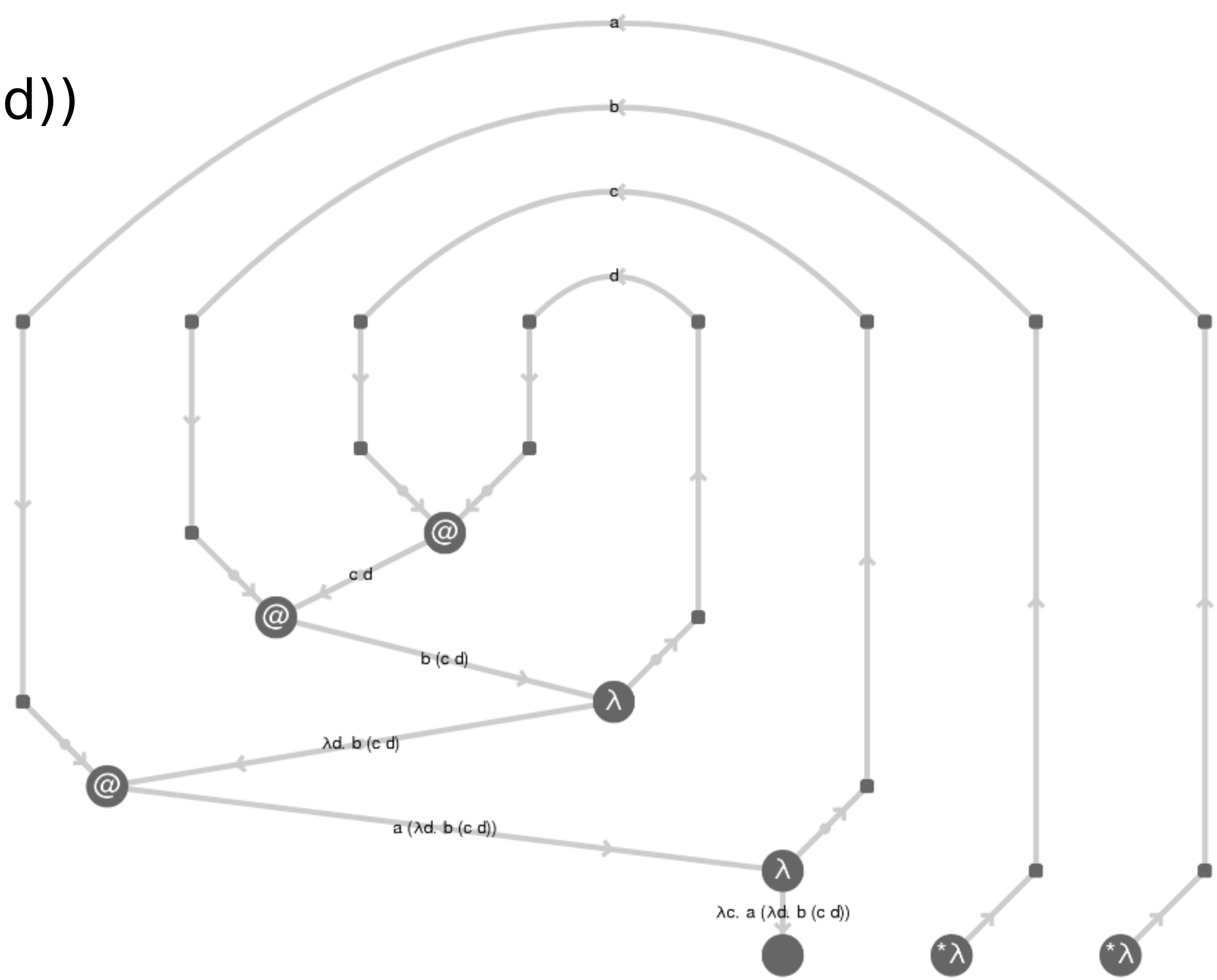
a,b ⊢ λc.a (λd.(b c) d)



*visualized with the help of https://www.georgejkaye.com/pages/fyp/visualiser.html

a 3-edge-connected term

a,b ⊢ λc.a (λd.b (c d))



*visualized with the help of https://www.georgejkaye.com/pages/fyp/visualiser.html

towards a general definition

A **cut** is a decomposition

$$t = C\{u\}$$

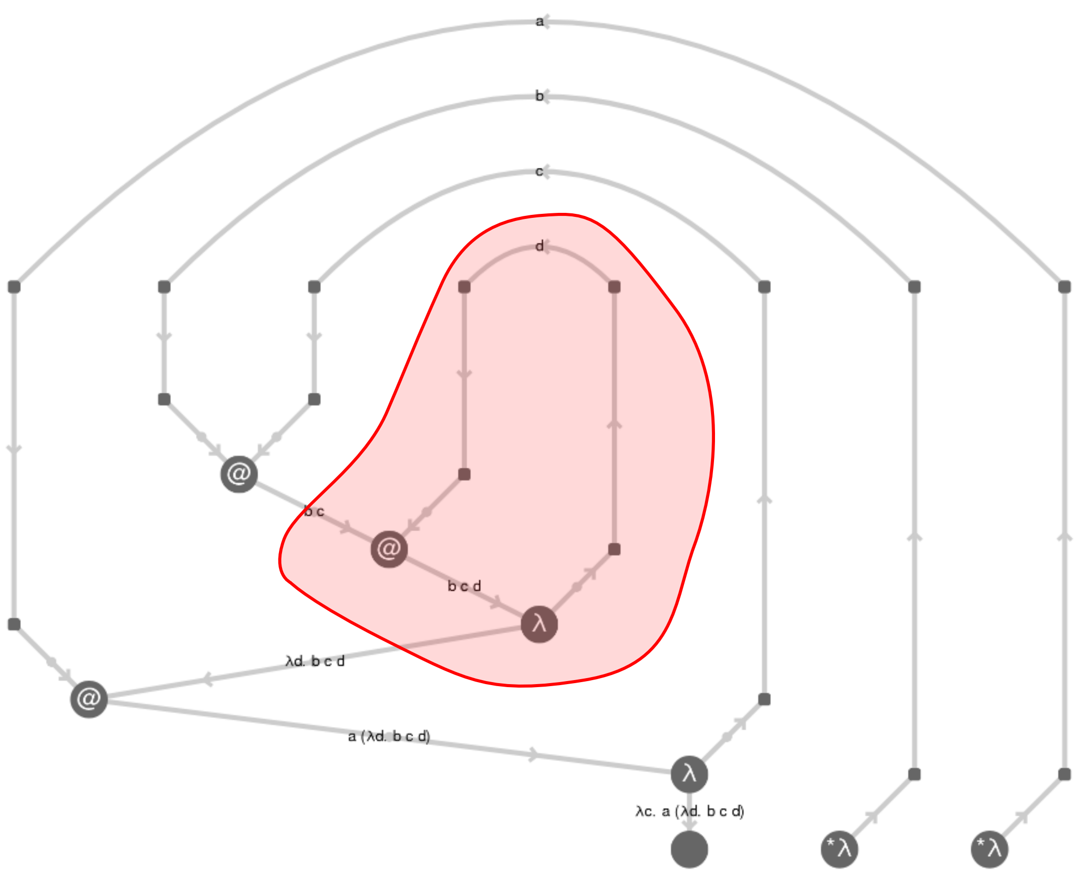
of a term t into a *subterm* u together with its surrounding *context* C.
Roughly speaking, a "context" is just a term with a hole/metavariable.

This definition gets a lot more interesting if we represent terms using HOAS and allow subterms to have higher type.

We say that the **type** of a cut $t = C\{u\}$ is the type of u.

towards a general definition

For example, a few slides ago, we saw a term with a cut of type U ⊸ U



a,b ⊢ λc.a (λd.b (c d))

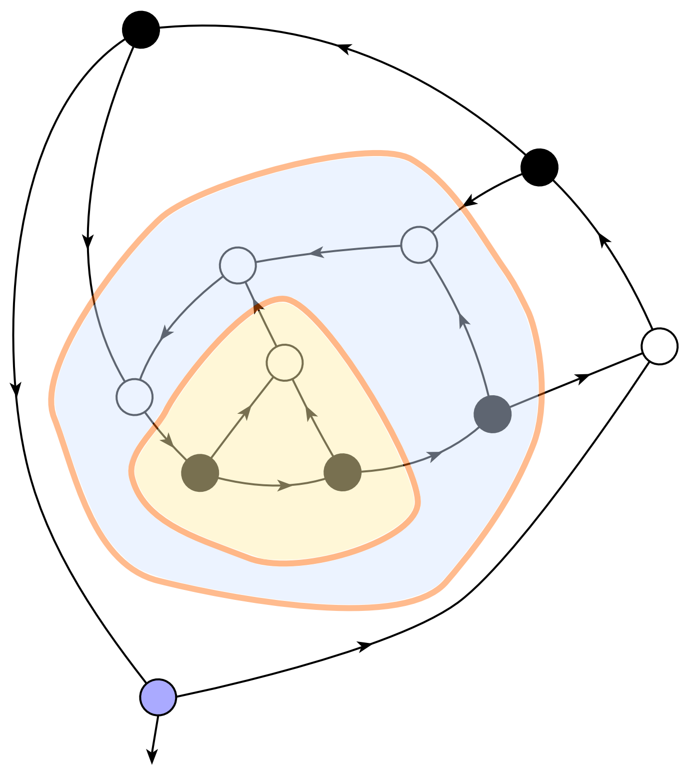
t : U ⊸ (U ⊸ U)
t = λa.λb.lam λc.app a (lam λd.app (app b c) d)

u : U ⊸ U
u = λx.lam λd.app x d

C : (U ⊸ U) ⇒ (U ⊸ (U ⊸ U))
C = {X}λa.λb.lam λc.app a (X (app b c))

towards a general definition

Here is an example of a term with a yellow cut of type (U ⊸ U) ⊸ U
and a blue cut of type U ⊸ (U ⊸ U)



λa.λb.λc.a (λd.λe.λf.(b (c d)) (e f))

t : U
t = lam λa.lam λb.lam λc. app a (lam λd.lam λe.lam λf.
app (app b (app c d)) (app e f))

u_1 : (U ⊸ U) ⊸ U
u_1 = λG.lam λe.lam λf.G (app e f)

C_1 : (U ⊸ U) ⊸ U ⇒ U
C_1 = {X}lam λa.lam λb.lam λc.
app a (lam λd.
X (λy.app (app b (app c d)) y))

u_2 : U ⊸ (U ⊸ U)
u_2 = λb.λc.lam λd.lam λe.lam λf.
app (app b (app c d)) (app e f))

C_2 : U ⊸ (U ⊸ U) ⇒ U
C_2 = {X}lam λa.lam λb.lam λc. app a (X b c)

towards a general definition

A term is said to be **k-indecomposable** if it does not have any non-trivial τ-cuts where τ is a type with $j < k$ occurrences of "U".

A cut $t = C\{u\}$ is said to be **trivial** if either C is the identity or u is elementary.

The **elementary** terms are as follows:

$$\begin{aligned} \lambda x.x &: U \multimap U \\ \mathrm{app} &: U \multimap (U \multimap U) \\ \mathrm{lam} &: (U \multimap U) \multimap U \end{aligned}$$

Claim: *t is k-indecomposable iff t is internally k-edge-connected.*

results & questions

3-indecomposable planar terms are counted by A000260, which also counts β-normal 2-indecomposable (= unitless) planar terms. Indeed, 3-indecomposable planar terms admit a direct inductive characterization...

$$t ::= x \mid C\{t\}$$
$$C ::= \lambda x.C \mid \bullet\ u$$

isomorphic to a similar characterization of β-normal unitless planar terms.

Conjecture: β-normal 3-indecomposable planar terms are counted by A000257!

What about non-planar 3-indecomposable terms?

results & questions

Theorem (Tutte 1962): 4-indecomposable planar terms are counted by A000256

Q: Is there a direct inductive construction of 4-indecomposable planar terms?

Theorem (Whitney 1931): every 4-indecomposable planar terms has a Hamiltonian cycle on its faces

Q: Is there a λ-calculus proof of Whitney's theorem?