Almost Every Simply Typed λ-Term Has a Long β-reduction Sequence

Ryoma Sin'ya (Akita University)

Computational Logic and Applicatdions

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THE CONTENTS OF THIS TALK:

- [FoSSaCS2017]
R. Sin'ya, K. Asada, N. Kobayashi, T. Tsukada:
"Almost Every Simply Typed **λ***-Term Has a Long* **β***-Reduction Sequence"*, FoSSaCS 2017

- [LMCS2019]
K. Asada, N. Kobayashi, R. Sin'ya, T. Tsukada:
"Almost Every Simply Typed **λ***-Term Has a Long* **β***-Reduction Sequence"*, LMCS Vol. 15 Issue 1, 2019

MOTIVATION

- A simply-typed term can have a very long **β**-reduction sequence.
  - *k*-EXP in the size of terms of order *k* [Beckmann 2001].

$$0\text{-EXP}(n) = n$$

$$(m+1)\text{-EXP}(n) = 2^{m\text{-EXP}(n)}$$

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e.g. $$(Twice)^n \underbrace{Twice \cdots Twice}_{k-2 \text{ times}}(\lambda x.bxx)((\lambda x.x)c)$$

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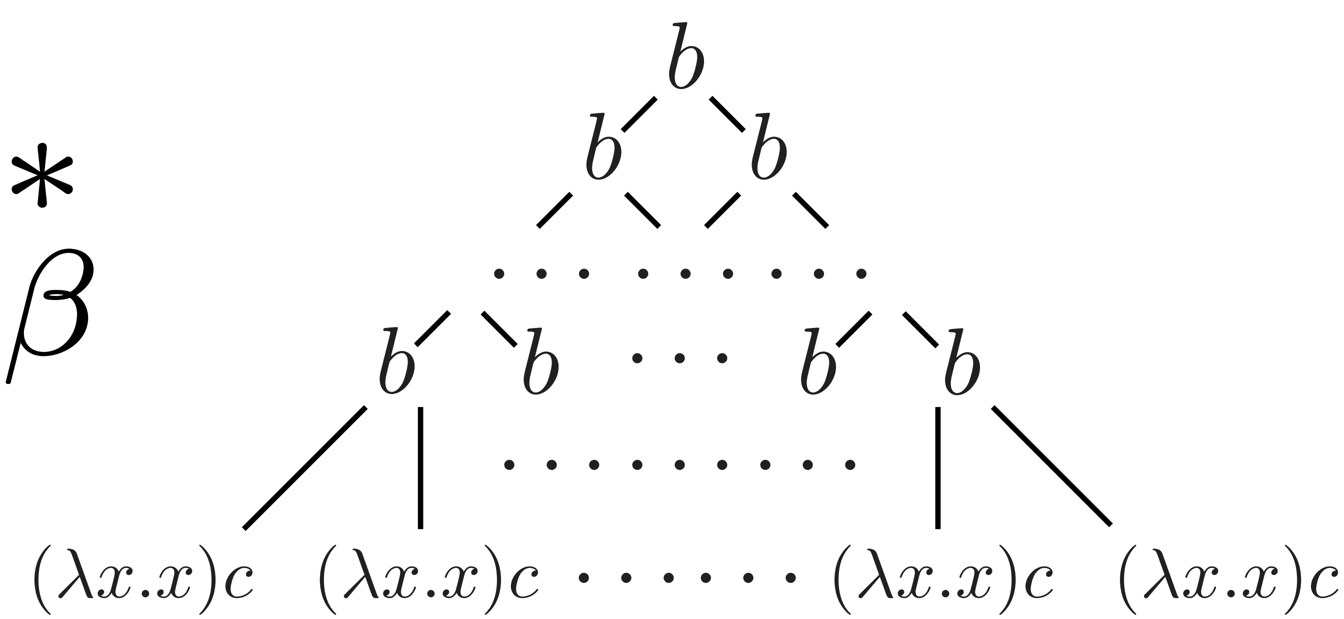

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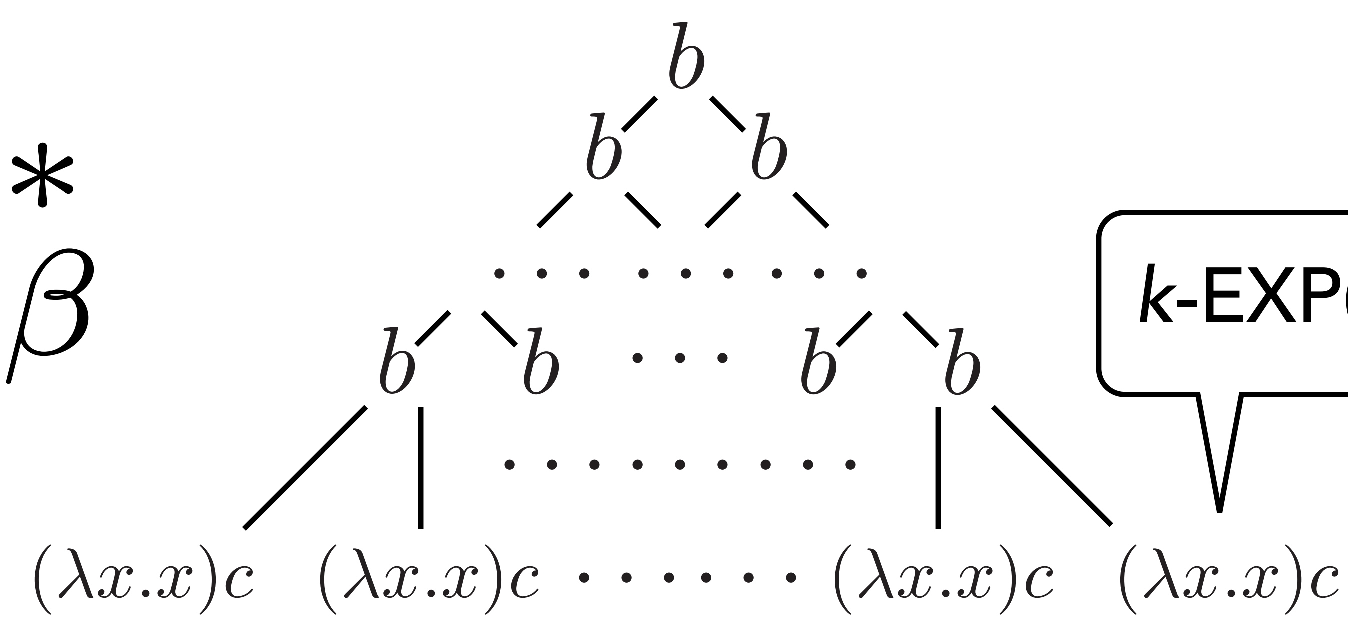

k-EXP(*n*) **β**-redexes of $(\lambda x.x)c$

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- How many terms have such long **β**-reduction sequences?

BACKGROUND

- The work has been motivated by quantitative analysis of the complexity of higher-order model checking (HOMC).

HIGHER-ORDER MODEL CHECKING [Ong 2006]

- Input : tree automaton $\mathcal{A}$ and **λ**Y-term *t*.
  Output : **YES** if $\mathcal{A}$ accepts the infinite tree represented by *t*, **NO** otherwise.
  Complexity: *k*-EXPTIME-complete for order-*k* **λ**Y-terms.

- We want to (*dis*)prove: HOMC can be efficiently solved for ***almost every input***.

OUTLINE

- Introduction
- Our result
- Proof of our result
- Related&future work
- Conclusion

OUR RESULT [FOSSACS2017]

For $k, \iota, \xi \geq 2$ *and* $k \leq \iota$,

$$\lim_{n\to\infty} \frac{\#\{[t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi)\} \mid \beta(t) \geq (k-2)\text{-EXP}(n)\}}{\#\Lambda_n^\alpha(k,\iota,\xi)} = 1.$$

$\Lambda_n^\alpha(k,\iota,\xi)$: the set of **α**-equivalence classes of size-*n* terms such that:

(1) the *order* is at most k.

(2) the *number of arguments* (*internal arity*) is at most ι.

(3) the number of *distinct variables* is at most ξ.

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the maximum length of **β**-reduction sequences of *t*.

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Almost every term of size *n* and order at most *k* has a **β**-reduction sequence of length (*k*-2)-EXP(*n*).

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NUMBER OF DISTINCT VARIABLES

- $\#\mathbf{V}(t)$: the # of variables in *t* excluding **unused variables**.
- For an **α**-equivalence class $[t]_\alpha$,

$$\#\mathbf{V}_\alpha([t]_\alpha) \triangleq \min\{\#\mathbf{V}(t') \mid t' \in [t]_\alpha\}$$

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<u>Example</u>

$$\#\mathbf{V}_\alpha([(\lambda z.z)\lambda y.x]_\alpha) = 1$$

$$\#\mathbf{V}((\underline{\lambda z.z})\lambda y.x) = \#\{x, z\} = 2$$

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OUR RESULT [FOSSACS2017]

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$$\lim_{n\to\infty} \frac{\#\{[t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi)\} \mid \beta(t) \geq (k-2)\text{-EXP}(n)\}}{\#\Lambda_n^\alpha(k,\iota,\xi)} = 1.$$

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$$\#\mathbf{V}_\alpha([t]_\alpha) \leq \xi \ \textit{ for every } \ [t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi)$$

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OVERVIEW OF OUR PROOF

- Almost every term contains a certain "context" that has a very long **β**-reduction sequence.

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- Almost every term contains a certain "context" that has a very long **β**-reduction sequence.

- Inspired by **Infinite Monkey Theorem**: for any word x, almost every word contains x as a subword.

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PROOF IDEA

1. Parameterising Infinite Monkey Theorem.

2. Extending (1) to **λ**-terms.

3. Constructing "explosive context" that generates a long **β**-reduction sequence.

MONKEY THEOREM

For any word x *over an alphabet A,*

$$\lim_{n\to\infty} \frac{\#\{w \in A^n \mid x \sqsubseteq w\}}{\#A^n} = 1.$$

$x \sqsubseteq w \;\Leftrightarrow\; w = uxv$ for some words $u, v \in A^*$.

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IDEA1: **PARAMETERISING** MONKEY THEOREM

For any family of words $(x_n)_n$ *over A such that*
$|x_n| = \lceil \log^{(2)}(n) \rceil,$

$$\lim_{n\to\infty} \frac{\#\{w \in A^n \mid x_n \sqsubseteq w\}}{\#A^n} = 1.$$

$\log^{(2)}(n) = \log(\log(n))$

IDEA2: EXTENDING IDEA1 TO **TERMS**

For any family of contexts $(C_n)_n$ *such that*
$|C_n| = \lceil \log^{(2)}(n) \rceil$,

$$\lim_{n\to\infty} \frac{\#\{[t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi)\} \mid C_n \preceq t\}}{\#\Lambda_n^\alpha(k,\iota,\xi)} = 1$$

if $k, \iota, \xi \geq 2$.

$C \preceq t \iff t = C'[C[t']]$ for some context C' and term t'.

IDEA3: CONSTRUCTING "**EXPLOSIVE**" CONTEXT

- For parameters *n* and *k*, we define the explosive context 💣$^{k}_{n}$ of order-*k* as:

$$\lambda x.\Big(\big(Twice\big)^n \underbrace{Twice \cdots Twice}_{k-2 \text{ times}} Dup\big(Id\,[\,]\big)\Big)$$

$$\text{where} \quad Twice = \lambda f.\lambda x.f(f\,x)$$

$$Dup = \lambda x.(\lambda y.\lambda z.y)xx \quad \text{and} \quad Id = \lambda x.x$$

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- It has the following "explosive property":

$$\beta\left(💣^{k}_{n}\right) \geq k\text{-EXP(n)}$$

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- It has the following "explosive property":

$$💣^k_n \preceq t \Rightarrow k\text{-EXP}(n) \leq \beta(t).$$

HARVEST

For $k, \iota, \xi \geq 2$ *and* $k \leq \iota$,

$$\lim_{n\to\infty} \frac{\#\{[t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi)\} \mid \text{💣}^k_{\lceil \log^{(2)}(n) \rceil} \preceq t\}}{\#\Lambda_n^\alpha(k,\iota,\xi)} = 1.$$

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For $k, \iota, \xi \geq 2$ *and* $k \leq \iota$,

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$\Downarrow$ A direct corollary of the explosive property:

$$\text{💣}^{k}_{\lceil \log^{(2)}(n) \rceil} \preceq t \Rightarrow (k-2)\text{-EXP}(n) \leq \beta(t).$$

Almost every term of size *n* and order at most *k* has a **β**-reduction sequence of length *(k-2)-EXP(n)*.

PROOF IDEA

1. Parameterising Infinite Monkey Theorem.

2. Extending (1) to **λ**-terms.

3. Constructing "explosive context" that generates a long **β**-reduction sequence.

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1. Parameterising Infinite Monkey Theorem.

2. Extending (1) to **λ**-terms.

Most technical part

3. Constructing "explosive context" that generates a long **β**-reduction sequence.

PROOF IDEA

1. Parameterising Infinite Monkey Theorem.

We first give a proof of (1), because it **clarify** the overall structure of the proof of (2).

2. Extending (1) to **λ**-terms.

Most technical part

3. Constructing "explosive context" that generates a long **β**-reduction sequence.

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http://en.wikipedia.org/wiki/Infinite_monkey_theorem

PROOF OF MONKEY THEOREM FOR WORDS

For any word x *over an alphabet A,*

$$\lim_{n\to\infty} \frac{\#\{w \in A^n \mid x \sqsubseteq w\}}{\#A^n} = 1.$$

$\because$

It suffice to show that:

$$\frac{\#\{w \in A^n \mid x \not\sqsubseteq w\}}{\#A^n} \to 0 \ (n \to \infty)$$

PROOF OF MONKEY THEOREM FOR WORDS

$\because$

$$\frac{\#\{w \in A^n \mid x \not\sqsubseteq w\}}{\#A^n} \stackrel{?}{\to} 0 \ (n \to \infty)$$

PROOF OF MONKEY THEOREM FOR WORDS

$\because$ Let $\ell = |x|, w \in A^n$.

$$\frac{\#\{w \in A^n \mid x \not\sqsubseteq w\}}{\#A^n} \overset{?}{\to} 0 \ (n \to \infty)$$

PROOF OF MONKEY THEOREM FOR WORDS

$\because$ Let $\ell = |x|, w \in A^n$.

$$w = w_1\, w_2\, \cdots\, w_{\lfloor n/\ell \rfloor}\, w'$$

$$\frac{\#\{w \in A^n \mid x \not\sqsubseteq w\}}{\#A^n} \overset{?}{\to} 0\ (n \to \infty)$$

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$\because$ Let $\ell = |x|, w \in A^n$.

$$w = \boxed{w_1} w_2 \cdots w_{\lfloor n/\ell \rfloor} w'$$

$$\underbrace{}_{\ell}$$

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$$w = \underbrace{w_1}_{\ell}\underbrace{w_2}_{\ell} \cdots \underbrace{w_{\lfloor n/\ell \rfloor}}_{\ell}\overbrace{w'}^{(n \bmod \ell)<\ell}$$

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$$w = \underbrace{w_1}_{\ell}\underbrace{w_2}_{\ell} \cdots \underbrace{w_{\lfloor n/\ell \rfloor}}_{\ell}\overbrace{w'}^{(n \bmod \ell)<\ell}$$

$$\frac{\#\{w \in A^n \mid x \not\sqsubseteq w\}}{\#A^n}$$

$$\leq \frac{\#\{w \in A^n \mid w_i \neq x \text{ for all } i \leq \lfloor n/\ell \rfloor\}}{\#A^n}$$

PROOF OF MONKEY THEOREM FOR WORDS

$\because$ Let $\ell = |x|, w \in A^n$.

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$$\frac{\#\{w \in A^n \mid x \not\sqsubseteq w\}}{\#A^n}$$

$$\leq \frac{\#\{w \in A^n \mid w_i \neq x \text{ for all } i \leq \lfloor n/\ell \rfloor\}}{\#A^n}$$

$$= \left(1 - \frac{1}{\#A^\ell}\right)^{\lfloor n/\ell \rfloor}$$

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$\because$ Let $\ell = |x|, w \in A^n$.

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DECOMPOSITION OF WORDS

cf.

$$A^n \ni w = \underbrace{w_1}_{\ell}\underbrace{w_2}_{\ell}\cdots\underbrace{w_{\lfloor n/\ell \rfloor}}_{\ell}\overbrace{w'}^{(n \bmod \ell)<\ell}$$

- Previous proof is based on a **"good"** decomposition of words.
  - This good decomposition is induced by the following coproduct-product form:

$$A^n \cong \coprod_{w' \in A^{(n \bmod \ell)}} \prod_{i \leq \lfloor n/\ell \rfloor} A^\ell$$

  - This point of view forms the basis of the later extensions.

PROOF OF PARAMETERISED MONKEY THEOREM FOR WORDS

For any family of words $(x_n)_n$ *over A such that* $|x_n| = \lceil \log^{(2)}(n) \rceil$,

$$\lim_{n \to \infty} \frac{\#\{w \in A^n \mid x_n \sqsubseteq w\}}{\#A^n} = 1.$$

$$\because \frac{\#\{w \in A^n \mid x_n \not\sqsubseteq w\}}{\#A^n}$$

$$\leq \frac{\#\{w \in A^n \mid \text{every decomposed part} \neq x\}}{\#A^n}$$

$$= \left(1 - \frac{1}{A^{\lceil \log^{(2)}(n) \rceil}}\right)^{\lfloor n / \lceil \log^{(2)}(n) \rceil \rfloor} \to 0 \ (n \to \infty) \qquad \therefore$$

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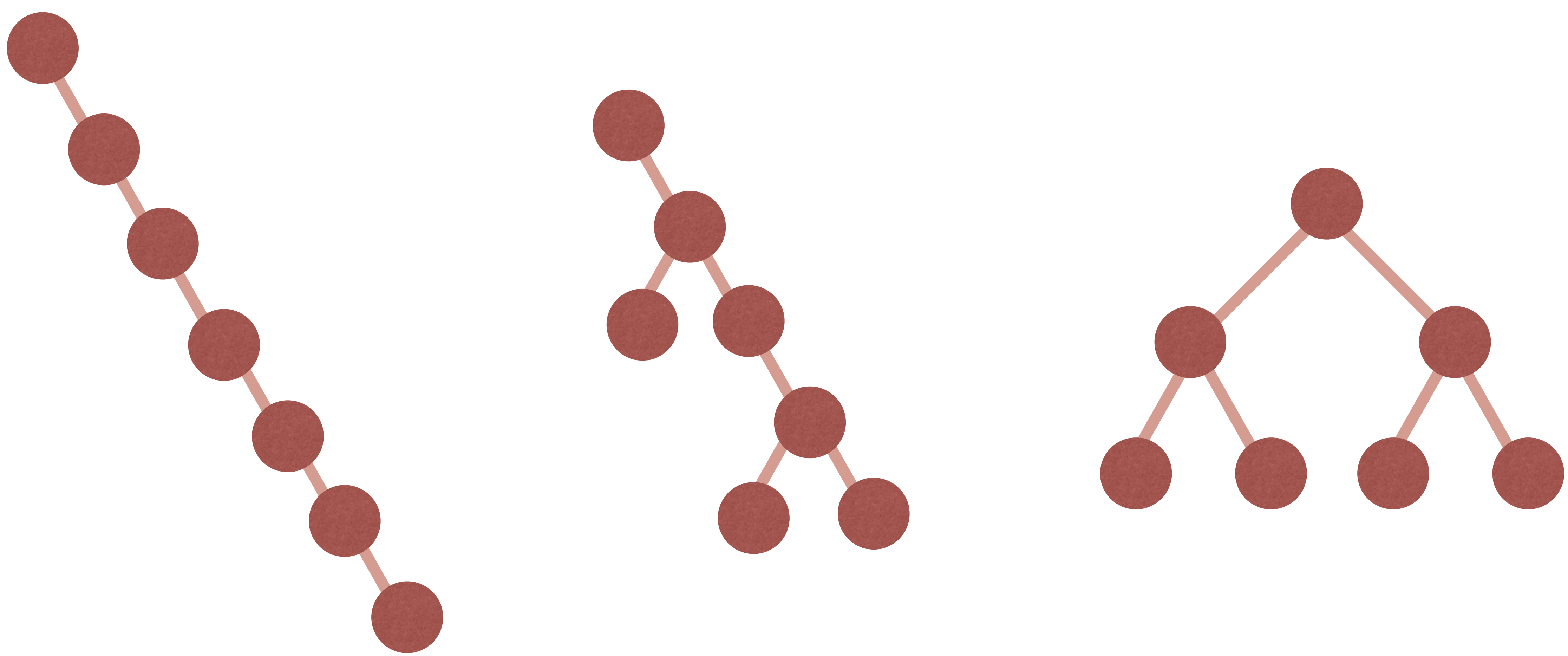
CHALLENGE IN PROVING PARAMETERISED MONKEY THEOREM FOR TERMS

- How to obtain such a "good" decomposition for the set of **λ**-terms $\Lambda_n^{\alpha}(k, \iota, \xi)$?

  - **Non-trivial** since terms have various shapes:



DECOMPOSITION OF TERMS

Example

$$\lambda x.(\lambda y.(\lambda z.\lambda x.z)(\lambda x.(\lambda x.y)\lambda z.z))x$$

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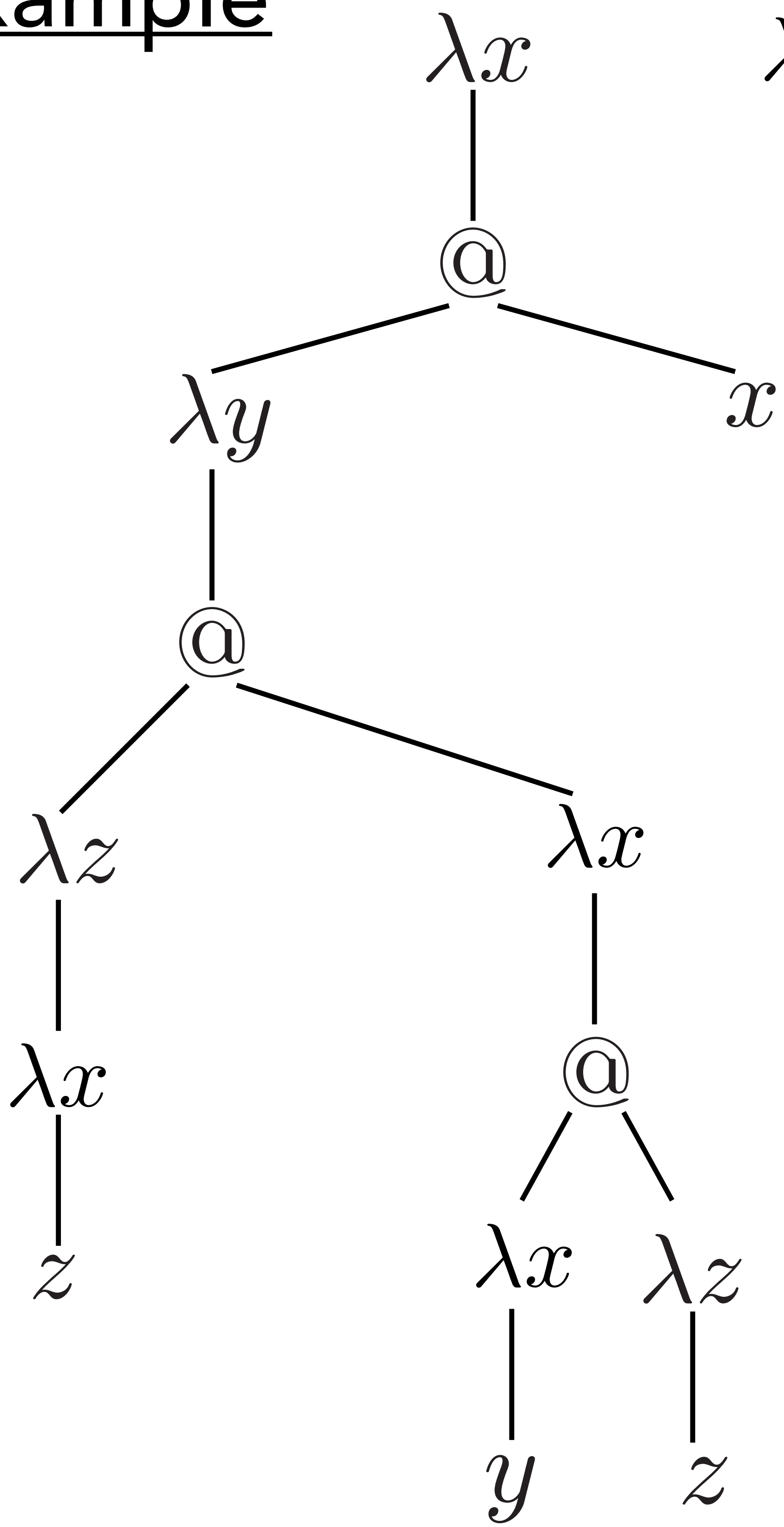

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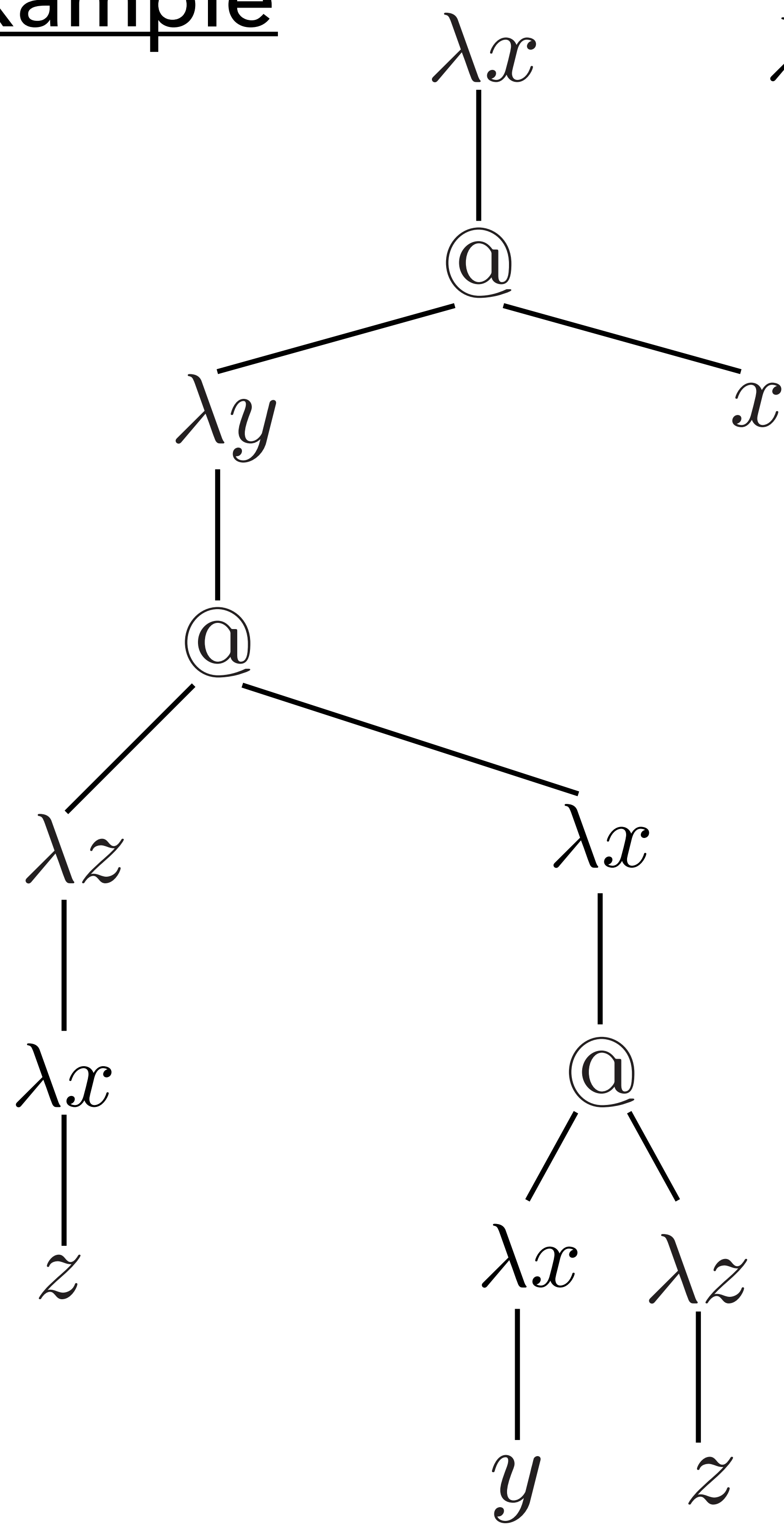

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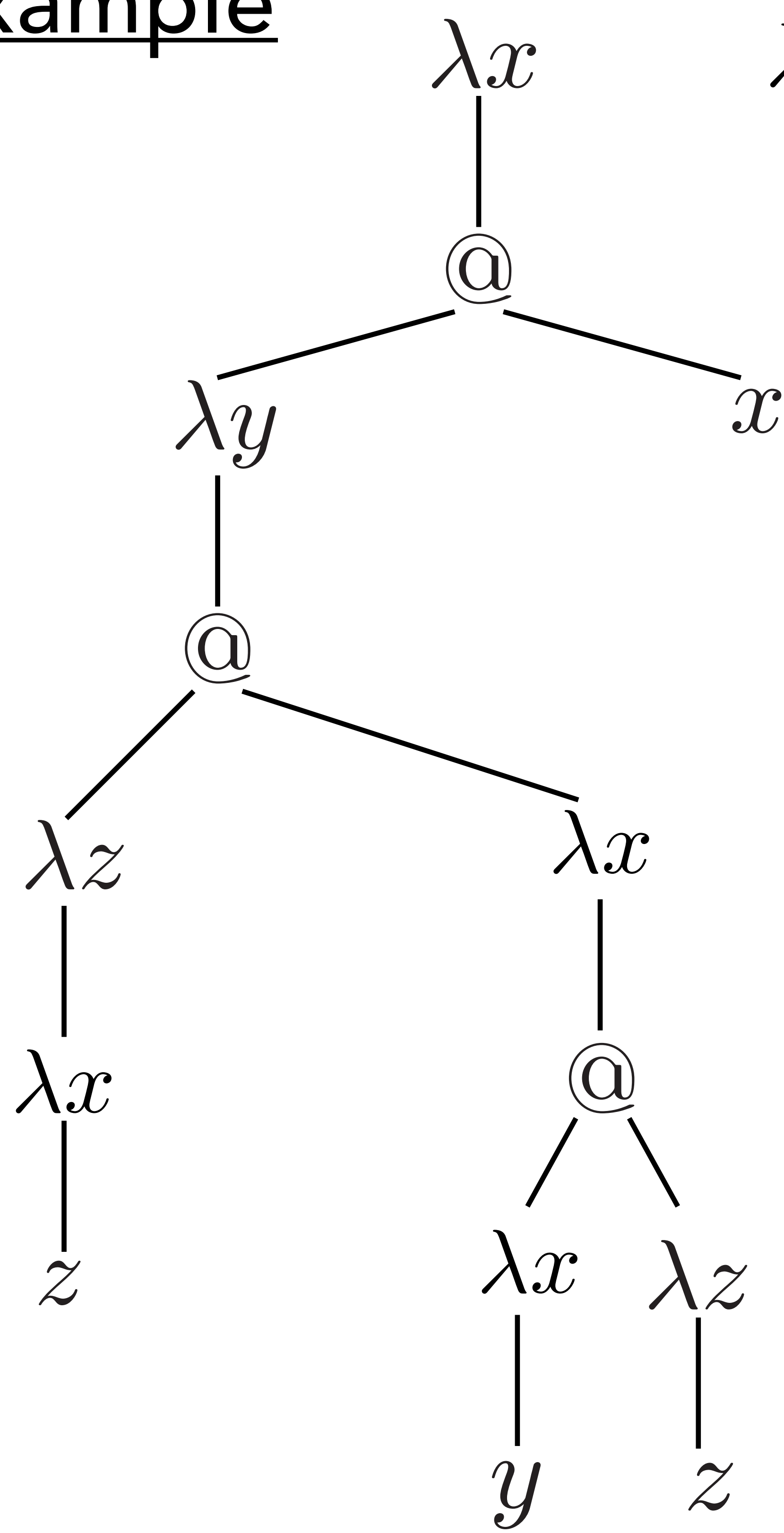

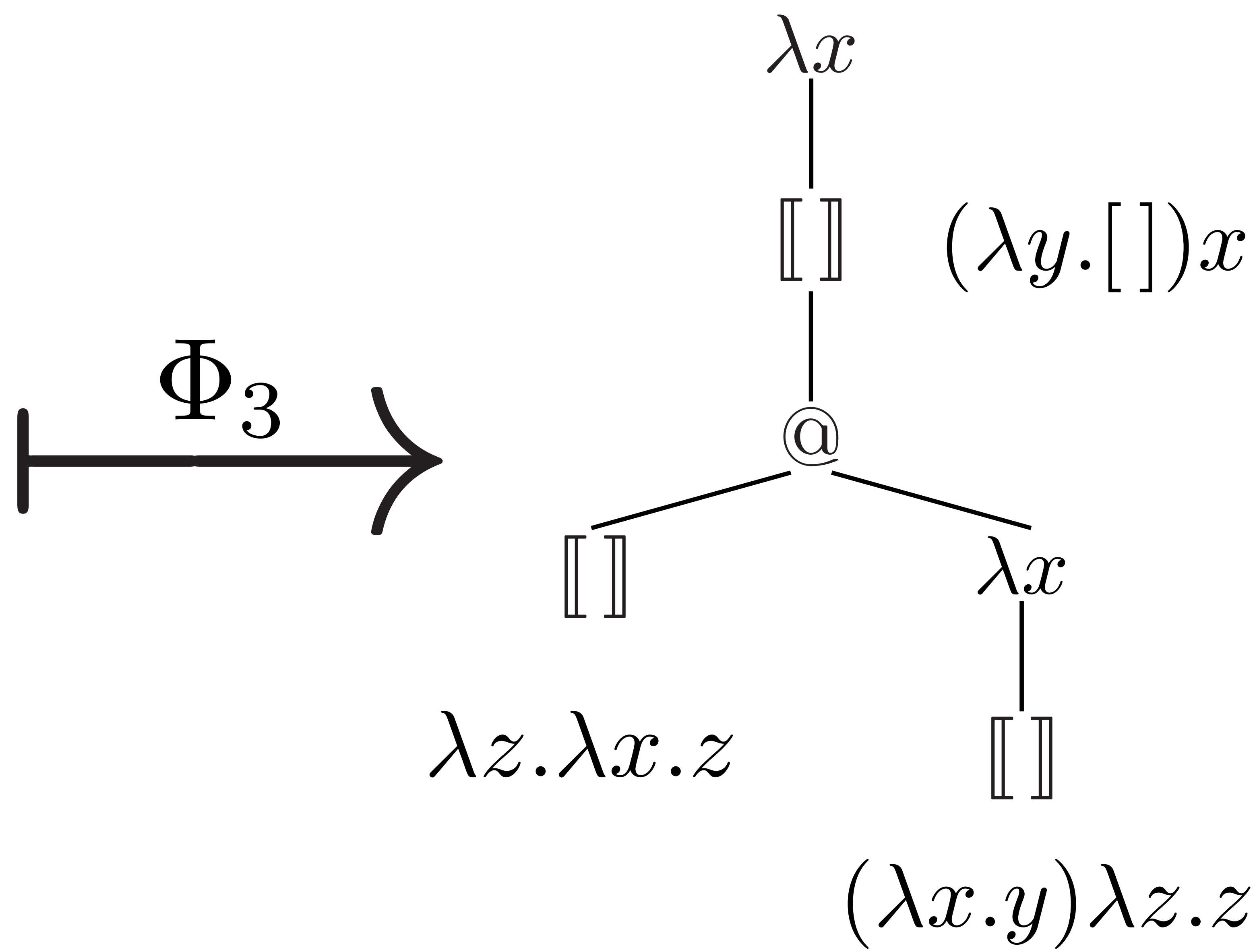

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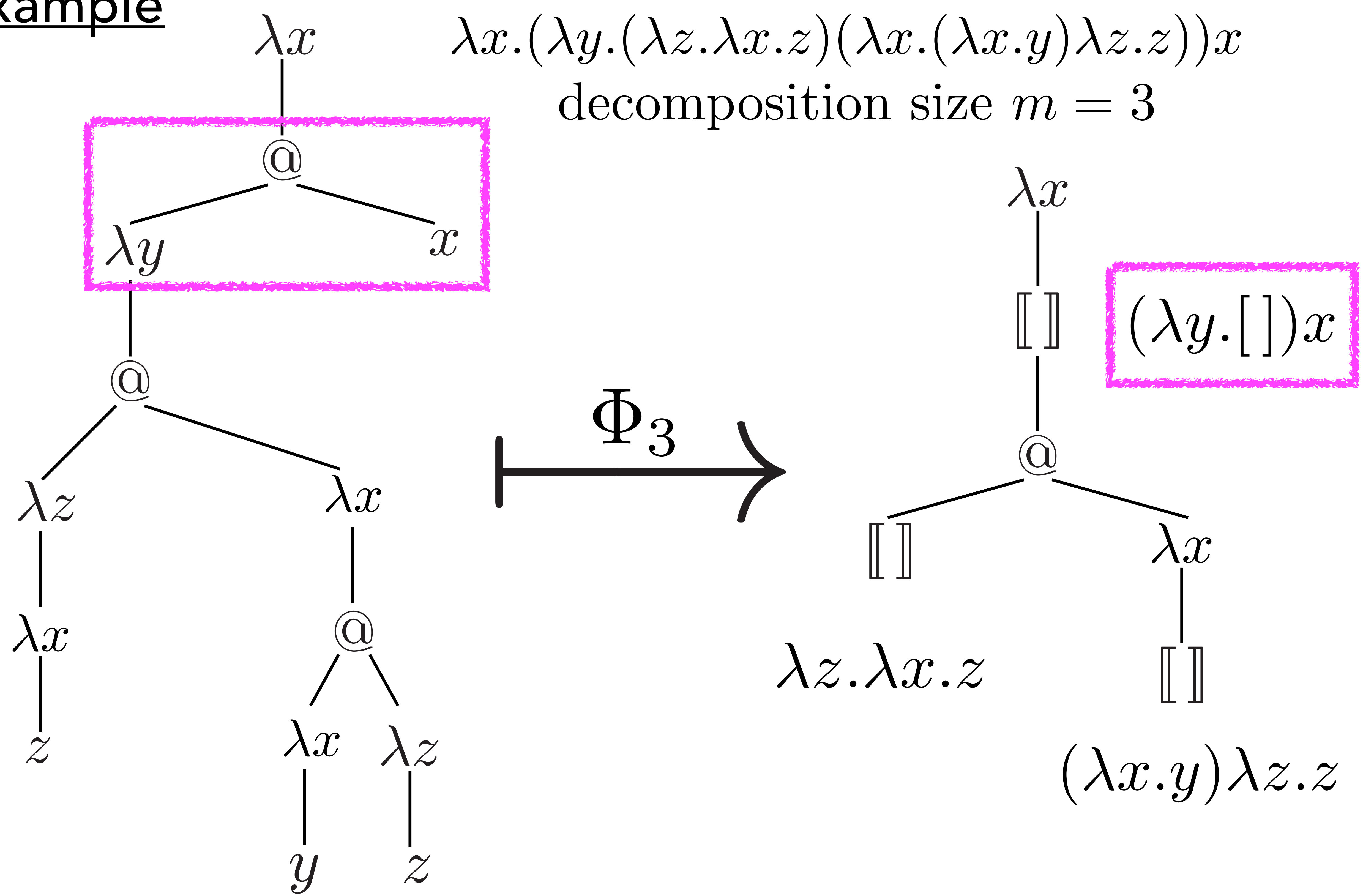

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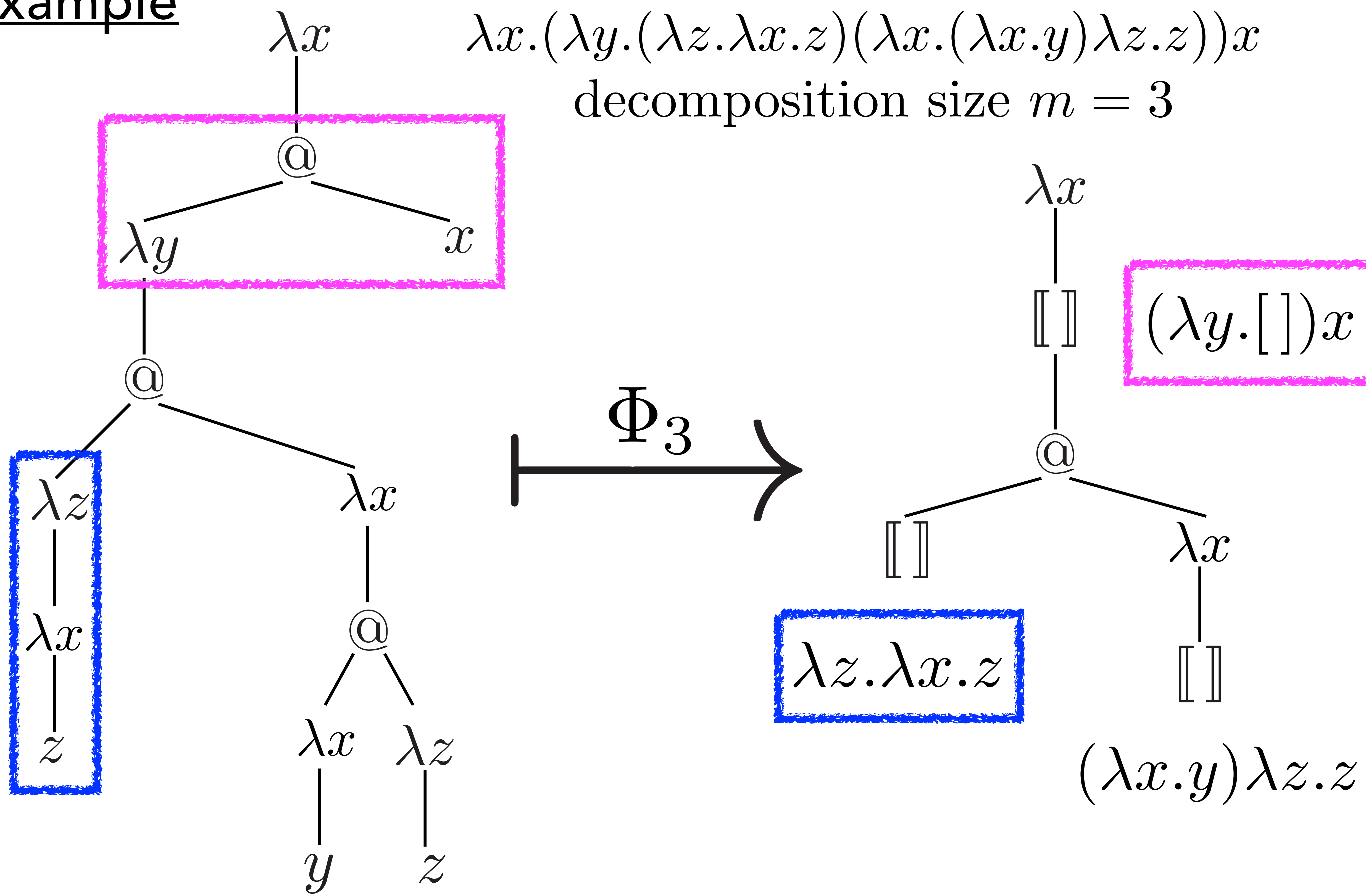

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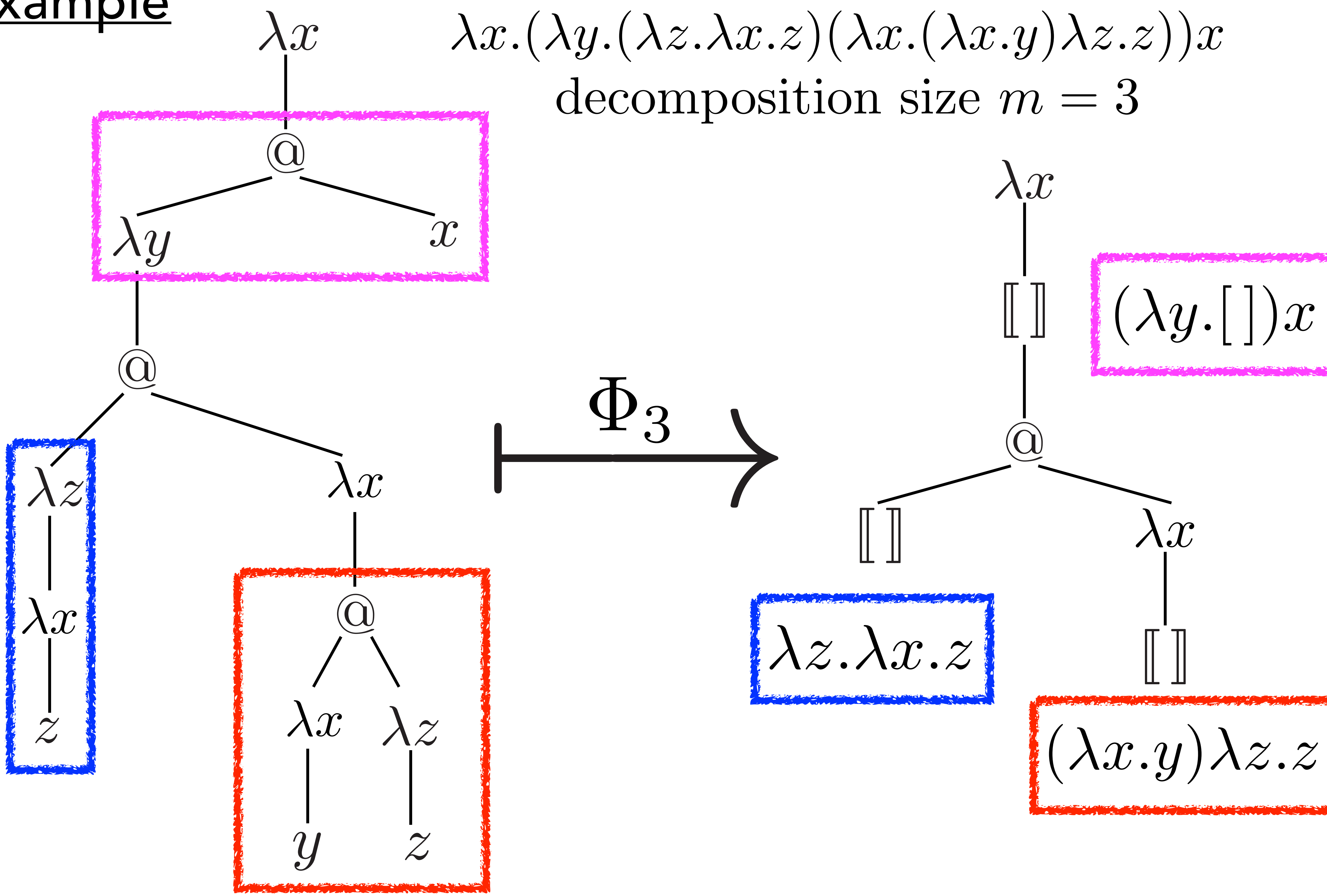
decomposition size $m = 3$

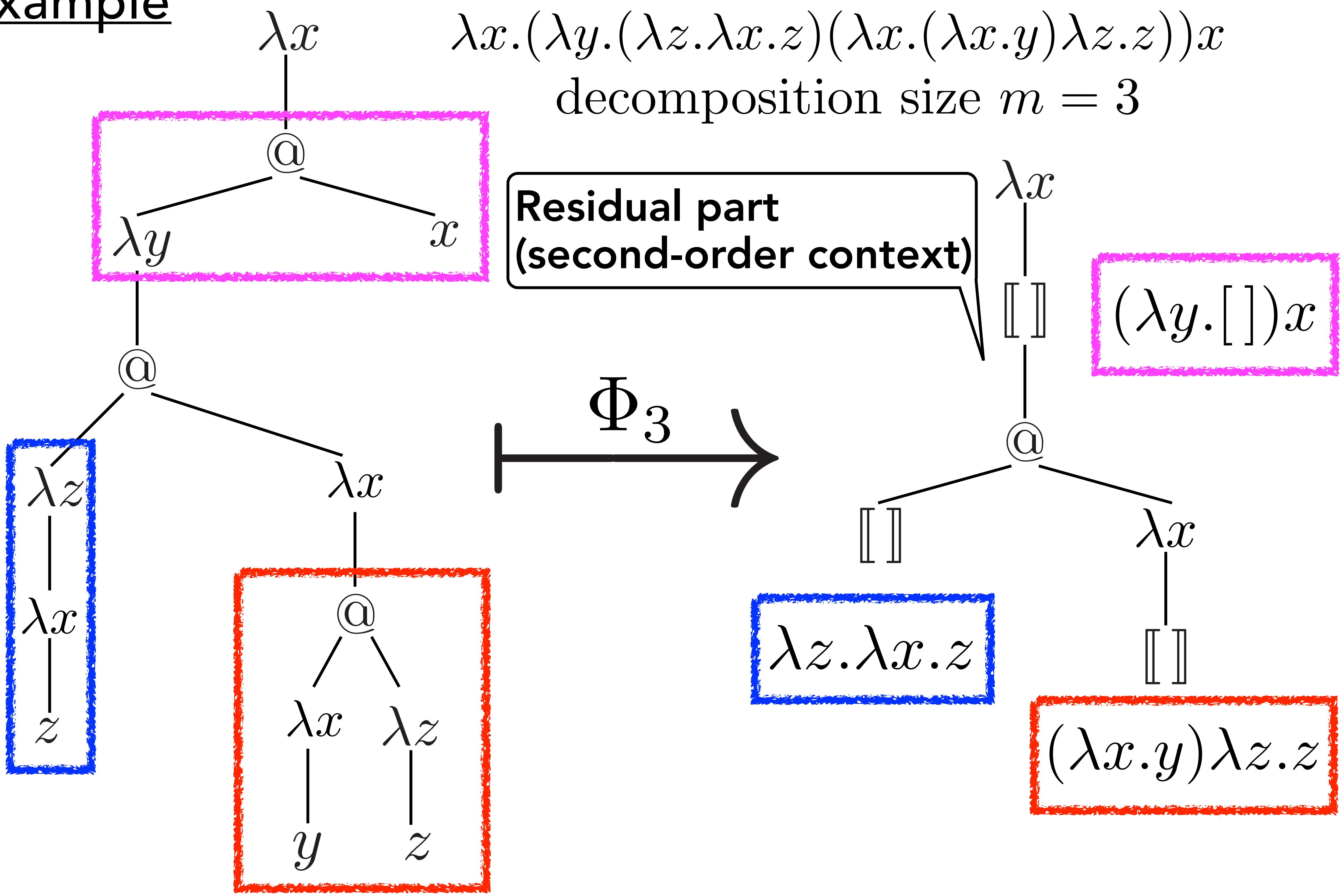


DECOMPOSITION OF TERMS

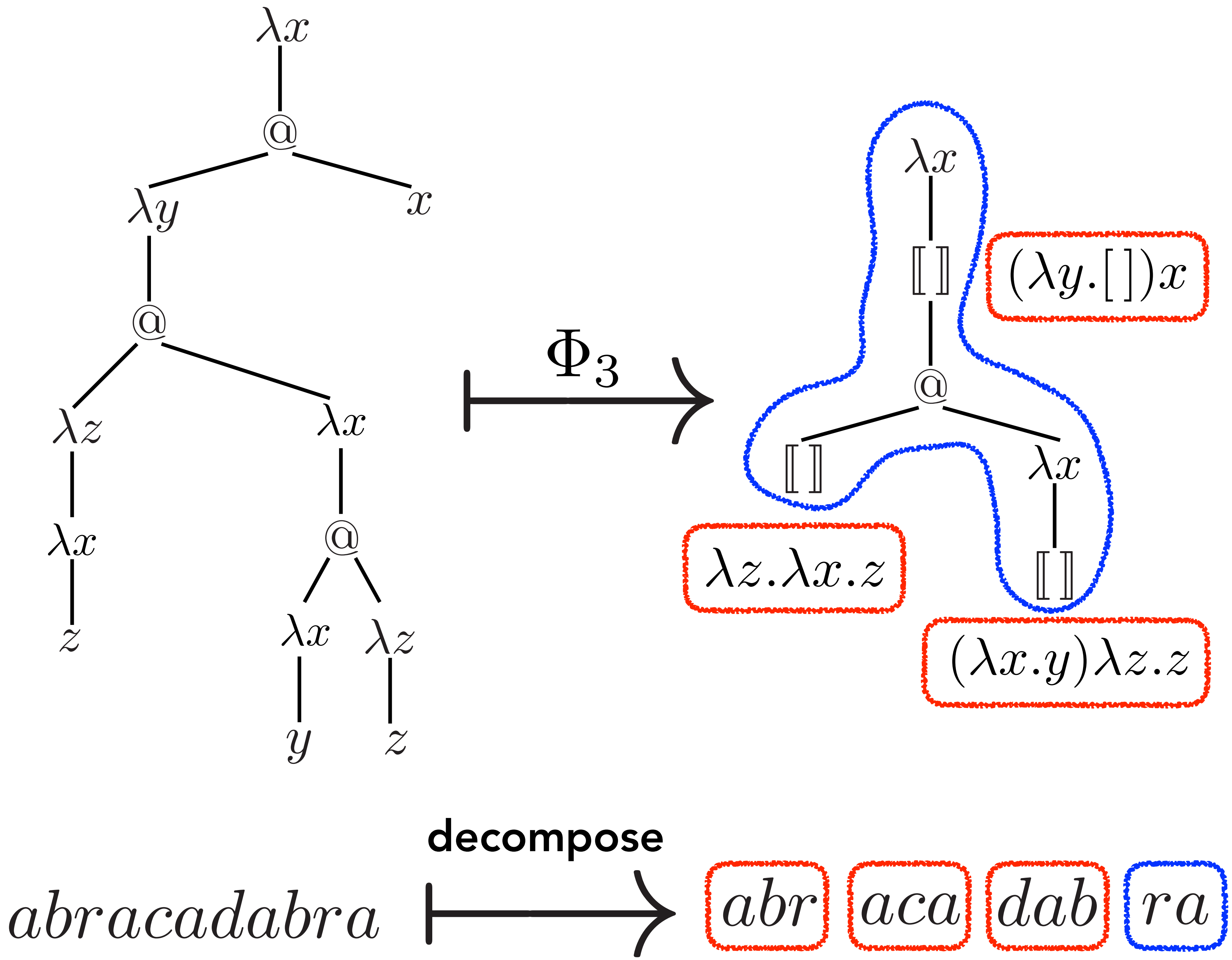
Example

$$\lambda x.(\lambda y.(\lambda z.\lambda x.z)(\lambda x.(\lambda x.y)\lambda z.z))x$$

decomposition size $m = 3$



ANALOGY BETWEEN THE DECOMPOSITION OF TERMS AND WORDS



* Decomposed part

* Residual part

A FORMAL DEFINITION OF THE DECOMPOSITION FUNCTION

If $|t| < m$, then $\varPhi_m(t) \triangleq (\llbracket\,\rrbracket, t, \epsilon)$.
If $|t| \geq m$, then:

$$\varPhi_m(\lambda\overline{x}^\tau.t_1) \triangleq \begin{cases} (E_1, \lambda\overline{x}^\tau.u_1, P_1) & \text{if } |\lambda\overline{x}^\tau.u_1| < m \\ (\llbracket\,\rrbracket[E_1], [\,], (\lambda\overline{x}^\tau.u_1)\cdot P_1) & \text{if } |\lambda\overline{x}^\tau.u_1| = m \end{cases}$$

$$\text{where } (E_1, u_1, P_1) = \varPhi_m(t_1).$$

$$\varPhi_m(t_1t_2) \triangleq \begin{cases} (\llbracket\,\rrbracket[(E_1\llbracket u_1\rrbracket)(E_2\llbracket u_2\rrbracket)], [\,], P_1\cdot P_2) & \text{if } |t_i| \geq m \ (i = 1, 2) \\ (E_1, u_1t_2, P_1) & \text{if } |t_1| \geq m, |t_2| < m, |u_1t_2| < m \\ (\llbracket\,\rrbracket[E_1], [\,], (u_1t_2)\cdot P_1) & \text{if } |t_1| \geq m, |t_2| < m, |u_1t_2| \geq m \\ (E_2, t_1u_2, P_2) & \text{if } |t_1| < m, |t_1u_2| < m \\ (\llbracket\,\rrbracket[E_2], [\,], (t_1u_2)\cdot P_2) & \text{if } |t_1| < m, |t_1u_2| \geq m \end{cases}$$

$$\text{where } (E_i, u_i, P_i) = \varPhi_m(t_i) \quad (i = 1, 2).$$

DECOMPOSITION LEMMA

For $k, \iota, \xi \geq 0$ *and* $n \geq m \geq 2$*,*

$$\Lambda^{\alpha}_{n}(k, \iota, \xi) \cong \coprod_{E \in \mathcal{B}^{n}_{m}} \prod_{i \leq \mathrm{shn}(E)} U^{m}_{E.i}$$

cf. $$A^{n} \cong \coprod_{w \in A^{(n \bmod m)}} \prod_{i \leq \lfloor \frac{n}{m} \rfloor} A^{m}$$

DECOMPOSITION LEMMA

For $k, \iota, \xi \geq 0$ *and* $n \geq m \geq 2$,

$$\Lambda_n^{\alpha}(k, \iota, \xi) \cong \coprod_{E \in \mathcal{B}_m^n} \prod_{i \leq \mathrm{shn}(E)} U_{E.i}^m$$

some set of second-order contexts

cf. $$A^n \cong \coprod_{w \in A^{(n \bmod m)}} \prod_{i \leq \lfloor \frac{n}{m} \rfloor} A^m$$

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some set of second-order contexts

the number of holes $[\![\,]\!]$ *in* E

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DECOMPOSITION LEMMA

For $k, \iota, \xi \geq 0$ *and* $n \geq m \geq 2$,

$$\Lambda^{\alpha}_{n}(k, \iota, \xi) \cong \coprod_{E \in \mathcal{B}^n_m} \prod_{i \leq \mathrm{shn}(E)} U^m_{E.i}$$

some set of second-order contexts

the number of holes $[\![\,]\!]$ *in E*

the set of "good" contexts that can be filled in the i-th hole of E.

cf. $$A^n \cong \coprod_{w \in A^{(n \bmod m)}} \prod_{i \leq \lfloor \frac{n}{m} \rfloor} A^m$$

DECOMPOSITION LEMMA

For $k, \iota, \xi \geq 0$ *and* $n \geq m \geq 2$ *,*

$$\Lambda_n^\alpha(k, \iota, \xi) \cong \coprod_{E \in \mathcal{B}_m^n} \prod_{i \leq \mathrm{shn}(E)} U_{E.i}^m$$

Each decomposed part $U_{E.i}^m$ **DOES depend** on the residual part E (and also on the index i)

Each decomposed part A^m **does NOT depend** on the residual part w

cf. $$A^n \cong \coprod_{w \in A^{(n \bmod m)}} \prod_{i \leq \lfloor \frac{n}{m} \rfloor} A^m$$

OUTLINE

- Introduction
- Our result
- Proof of our result
  - Idea
  - Infinite Monkey Theorem
  - Decomposition of terms
  - Sketch of the proof
- Related&future work
- Conclusion

PROOF OF PARAMETERISED MONKEY THOREM FOR TERMS

For any family of contexts $(C_n)_n$ *of* $\Lambda_n^\alpha(k,\iota,\xi)$ *such that* $|C_n| = \lceil \log^{(2)}(n) \rceil$,

$$\lim_{n\to\infty} \frac{\#\{[t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi))\} \mid C_n \preceq t\}}{\#\Lambda_n^\alpha(k,\iota,\xi)} = 1.$$

if $k,\iota,\xi \geq 2$.

$\because$ It is suffice to show that

$$\frac{\#\{[t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi) \mid C_n \not\preceq t\}}{\#\Lambda_n^\alpha(k,\iota,\xi)} \to 0 \quad (n \to \infty)$$

PROOF OF PARAMETERISED MONKEY THOREM FOR TERMS

$\because$

$$\frac{\#\{[t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi) \mid C_n \not\leq t\}}{\#\Lambda_n^\alpha(k,\iota,\xi)} \overset{?}{\to} 0 \quad (n \to \infty)$$

cf. $\Lambda_n^\alpha(k,\iota,\xi) \cong \coprod_{E \in \mathcal{B}_n^{\log^{(2)}(n)}} \prod_{i \leq \mathrm{shn}(E)} U_{E.i}^{\log^{(2)}(n)}$

$$\begin{array}{ccc} \Lambda_n^\alpha(k,\iota,\xi) & \coprod_{E} & \prod_{i} U_{E.i} \\ \Cup & \Cup & \Cup \\ [t]_\alpha & \xmapsto{\Phi_{\log^{(2)}(n)}} E \;\&\; & (u_1, u_2, \cdots, u_{\mathrm{shn}(E)}) \end{array}$$

PROOF OF PARAMETERISED MONKEY THOREM FOR TERMS

$\because$

$$\frac{\#\{[t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi) \mid C_n \not\preceq t\}}{\#\Lambda_n^\alpha(k,\iota,\xi)}$$

$$\leq \frac{\#\{[t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi) \mid C_n \not\preceq u_i \text{ for every } i\}}{\#\Lambda_n^\alpha(k,\iota,\xi)}$$

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$\quad \Cup \qquad\qquad\qquad \Cup \qquad\qquad \Cup$

$$[t]_\alpha \xmapsto{\Phi_{\log^{(2)}(n)}} E \;\&\; (u_1, u_2, \cdots, u_{\mathrm{shn}(E)})$$

PROOF OF PARAMETERISED MONKEY THOREM FOR TERMS

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PROOF OF PARAMETERISED MONKEY THOREM FOR TERMS

$$\vdots$$

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$$\leq \left(1 - 1/c\gamma^{2\lceil \log^{(2)}(n) \rceil}\right)^{n/4\lceil \log^{(2)}(n) \rceil}$$

PROOF OF PARAMETERISED MONKEY THOREM FOR TERMS

...

Lemma

$\mathrm{shn}(E) \geq n/4\lceil \log^{(2)}(n) \rceil$ for any $E \in \mathcal{B}_n^{\lceil \log^{(2)}(n) \rceil}$

$$\leq \frac{\#\{[t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi) \mid C_n \not\leq u_i \text{ for every } i\}}{\#\Lambda_n^\alpha(k,\iota,\xi)}$$

$$= \frac{\# \coprod_{E \in \mathcal{B}_n^{\lceil \log^{(2)}(n) \rceil}} \prod_{i \leq \mathrm{shn}(E)} \left\{ u_i \in U_{E.i}^{\lceil \log^{(2)}(n) \rceil} \mid C_n \not\leq u_i \right\}}{\#\Lambda_n^\alpha(k,\iota,\xi)}$$

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PROOF OF PARAMETERISED MONKEY THOREM FOR TERMS

$\vdots$

Lemma

$\mathrm{shn}(E) \geq n/4\lceil\log^{(2)}(n)\rceil$ for any $E \in \mathcal{B}_n^{\lceil\log^{(2)}(n)\rceil}$

$$\leq \frac{\#\{[t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi) \mid C_n \not\leq u_i \text{ fo[r] every } i\}}{\ldots}$$

Lemma

$\#U_{E.i}^{\lceil\log^{(2)}(n)\rceil} = \mathrm{O}(c\gamma^{2\lceil\log^{(2)}(n)\rceil})$

for some constants c and γ

$$= \frac{\ldots (n)\rceil \mid C_n \not\leq u_i\}}{\#\Lambda_n^\alpha(k,\iota,\xi)}$$

$$\leq \left(1 - 1/c\gamma^{2\lceil\log^{(2)}(n)\rceil}\right)^{n/4\lceil\log^{(2)}(n)\rceil}$$

PROOF OF PARAMETERISED MONKEY THOREM FOR TERMS

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PROOF OF PARAMETERISED MONKEY THOREM FOR TERMS

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$$\leq \left(1 - 1/c\gamma^{2\lceil \log^{(2)}(n) \rceil}\right)^{n/4\lceil \log^{(2)}(n) \rceil} \to 0 \ (n \to \infty) \quad \therefore$$

PROOF OF PARAMETERISED MONKEY THOREM FOR TERMS

For any family of contexts $(C_n)_n$ *of* $\Lambda_n^\alpha(k,\iota,\xi)$ *such that* $|C_n| = \lceil \log^{(2)}(n) \rceil$,

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$$\frac{\#\{[t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi) \mid C_n \not\preceq t\}}{\#\Lambda_n^\alpha(k,\iota,\xi)} \to 0 \quad (n \to \infty)$$

$\therefore$

SUMMARY OF THE MAIN PROOF

the probability that a term $[t]_\alpha \in \Lambda_n^\alpha(k, \iota, \xi)$ has a β-reduction sequence of length (k-2)-EXP(n)

(∵ explosive property)

$\geq$ the probability that 💣$^{k}_{\lceil \log^{(2)}(n) \rceil} \preceq t$ *holds*

(∵ Patermeterised Monkey Theorem)

$\to 1 \quad (n \to \infty)$

OUTLINE

- Introduction
- Our result
- Proof of our result
- Related&future work
- Conclusion

OUR RECENT PAPER [LMCS2019]

- We have strengthened and generalised the result of [FoSSacs2017]
  - Strengthen: we prove that almost every **λ**-term of order-k has a (k-1)-EXP long **β**-reduction sequence.
  - Generalise: we prove the parameterised monkey theorem for trees generated by any (unambiguous and strongy-connected) regular tree grammar, not only for **λ**-terms **Λ***(k,* **ι***,* **ξ***)*.

RELATED WORK

- Quantitative analysis of untyped terms:
  - Almost every **λ**-term is strongly normalising (**SN**), but almost every SK-combinatory term is not **SN** [David *et al.* 2009].
  - Almost every de Bruijn **λ**-term is not **SN** [Bendkowski *et al.* 2015].
  - Empirical results: almost every **λ**-term is not **β**-normal, untypable [Grygiel-Lescanne 2013].

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  - Almost every de Bruijn **λ**-term is not **SN** [Bendkowski *et al.* 2015].
  - Empirical results: almost every **λ**-term is not **β**-normal, untypable [Grygiel-Lescanne 2013].
- Quantitative analysis of typed terms: little is known.

FUTURE WORK

- Quantitative analysis of simply typed **λ**-terms in different settings:
  - with an **unbounded** number of variables.
    - this makes # of terms super-exponential growth.

FUTURE WORK

- Quantitative analysis of simply typed **λ**-terms in different settings:
  - with an **unbounded** number of variables.
    - this makes # of terms super-exponential growth.
- Quantitative analysis of the complexity of HOMC:
  - We are trying to prove the following kind

$$\lim_{n\to\infty} \frac{\#\left(\left\{ [t]_\alpha \in \Lambda_n^Y \mid \mathrm{HOMC}(t, \cdot\,) \text{ is } k\text{-EXP-hard} \right\} \right)}{\#\left(\hat{\Lambda}_n^Y \right)} = 1.$$

for a certain fragment of **λ**Y-terms (on-going work).

CONCLUSION

- We want to know the typical-case complexity of HOMC.
  - Analysis of the length of **β**-reduction sequence of STLC as a first step.

- Result: almost every terms of order at most *k* has a (*k-1*)-exponentially long **β**-reduction sequence.
  - The core of our proof is a non-trivial extension of well-known Monkey Theorem.
  - The parameterised Monkey Theorem for regular tree languages may be of independent interest.


Thank you!

APPENDIX

UNUSED VARIABLE

- Our syntax has special symbol $*$, an unused variable:

$$t ::= x \mid \lambda x.t \mid \lambda *.t \mid tt$$

  - $*$ is never used, appeared only in **λ** binder.

  - **α**-equivalence is defined naturally.

<u>Example</u> $\lambda y.x \approx_\alpha \lambda *.x$

SUPER-EXP. GROWTH

- Quantitative analysis of simply typed **λ**-terms in different settings:
  - with an **unbounded** number of variables.
    - this makes # of terms super-exponential growth.
    - The number of term of the form below is *n!*:

$$\lambda x_1^{\circ \to \circ} \cdots x_n^{\circ \to \circ} a^{\circ} . x'_1 (x'_2 (\cdots (x'_n a) \cdots))$$

- In the fragment of **λ**-terms with super-exponential growth, (classical) Monkey Theorem *does not hold* (cf. David et al.).

ANALOGY BETWEEN DECOMPOSITION OF WORDS AND TERMS $(m = \lceil \log^{(2)}(n) \rceil)$

	Residual part	Decomposed part	# of decomposed parts	Size of each decomposed part
Words	$A^{n \bmod \lceil \log^{(2)}(n) \rceil}$	$A^{\lceil \log^{(2)}(n) \rceil}$	$\geq \dfrac{n}{\lceil \log^{(2)}(n) \rceil}$	$\#A^{\lceil \log^{(2)}(n) \rceil}$
Terms	$\mathcal{B}_n^{\lceil \log^{(2)}(n) \rceil}$	$U_{E.i}^{\lceil \log^{(2)}(n) \rceil}$	$\geq \dfrac{n}{4\lceil \log^{(2)}(n) \rceil}$	$\mathrm{O}(\gamma^{2\lceil \log^{(2)}(n) \rceil})$ for some γ

Some set of second-order contexts

The set of "good" contexts

Upper bound of $\#U_{E.i}^{\lceil \log^{(2)}(n) \rceil}$

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Upper bound of $\#U_{E.i}^{\lceil \log^{(2)}(n) \rceil}$

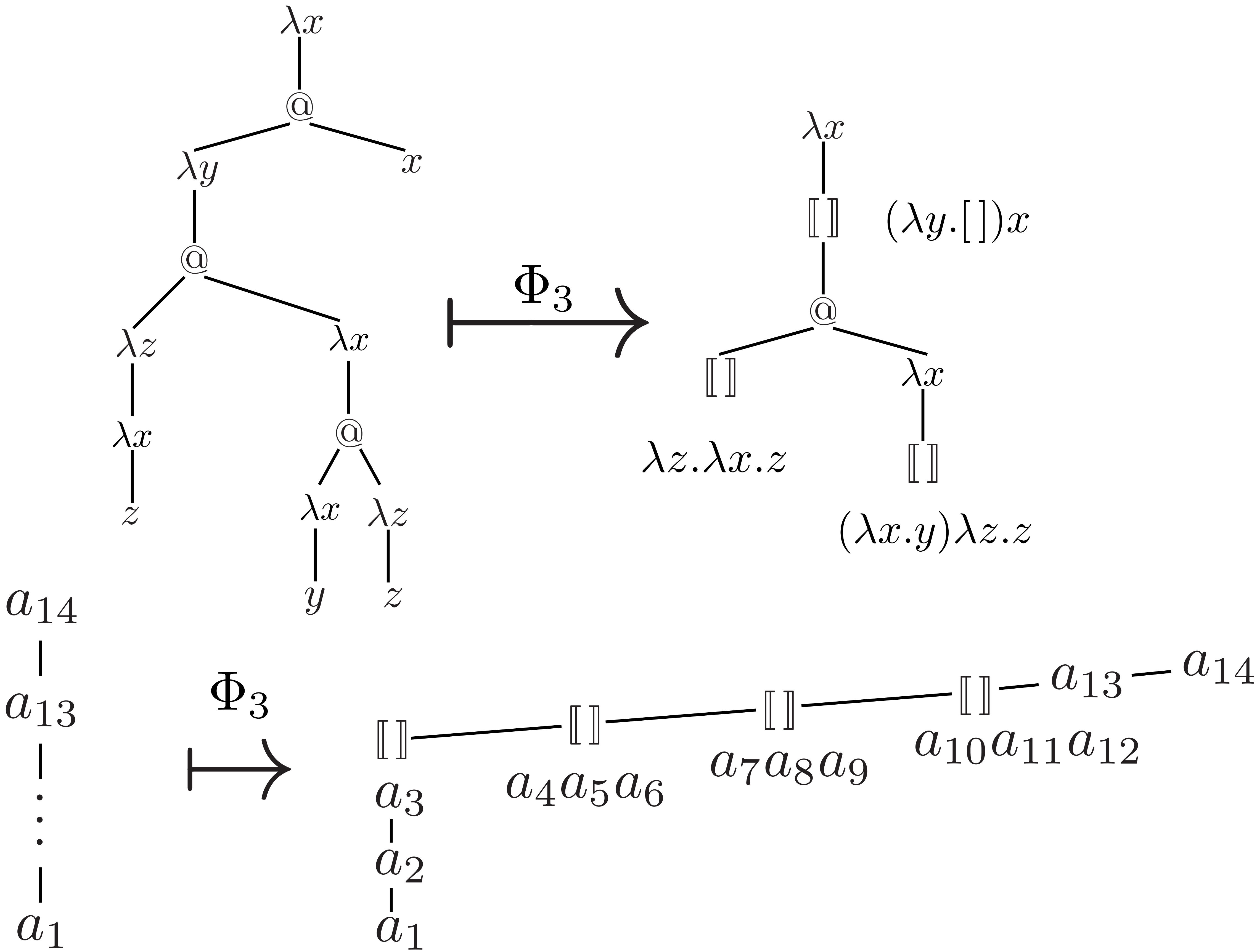
A FORMAL DEFINITION OF THE DECOMPOSITION FUNCTION

We define Φ_m by:

- If $|T| < m$, then $\Phi_m(T) \triangleq ([\,], T, \epsilon)$.
- If $|T| \geq m$, $T = a(T_1, \ldots, T_{\Sigma(a)})$, and $\Phi_m(T_i) = (U_i, E_i, P_i)$ (for each $i \leq \Sigma(a)$), then:

$$\Phi_m(T) \triangleq \begin{cases} \big([\,],\ a\big(U_1[E_1], \ldots, U_{\Sigma(a)}[E_{\Sigma(a)}]\big),\ P_1 \cdots P_{\Sigma(a)}\big) \\ \quad \text{if there exist } i, j \text{ such that } 1 \leq i < j \leq \Sigma(a) \text{ and } |T_i|, |T_j| \geq m \\ \big([\,],\ [\![\,]\!]_1^n[E_i],\ a(T_1, \ldots, U_i, \ldots, T_{\Sigma(a)}) \cdot P_i\big) \\ \quad \text{if } |T_j| < m \text{ for every } j \neq i,\ |T_i| \geq m, \text{ and} \\ \qquad n \triangleq |a(T_1, \ldots, U_i, \ldots, T_{\Sigma(a)})| \geq m \\ \big(a(T_1, \ldots, U_i, \ldots, T_{\Sigma(a)}),\ E_i,\ P_i\big) \\ \quad \text{if } |T_j| < m \text{ for every } j \neq i,\ |T_i| \geq m, \text{ and} \\ \qquad |a(T_1, \ldots, U_i, \ldots, T_{\Sigma(a)})| < m \\ \big([\,],\ [\![\,]\!]_0^n,\ T\big) \\ \quad \text{if } |T_i| < m \text{ for every } i \leq \Sigma(a), \text{ and } n \triangleq |T| \end{cases} \quad (3.$$

RELATION BETWEEN THE DECOMPOSITION OF TERMS AND WORDS



PROOF OF MONKEY THEOREM (FORMAL)

$\because$ Let $\ell = |x|$.

$$\Pr[x \not\sqsubseteq w \text{ holds for a randomly chosen word } w \in A^n]$$

$$\leq \Pr[x \neq w_i \text{ holds for every decomposed part } w_i \text{ of } w]$$

$$= \frac{\#\left(\coprod_{w' \in A^{(n \bmod \ell)}} \prod_{i \leq \lfloor n/\ell \rfloor} (A^\ell \setminus \{x\})\right)}{\#A^n}$$

ページ移動で式変形が追えない

$$= \frac{\sum_{w' \in A^{(n \bmod \ell)}} \#\left(\prod_{i \leq \lfloor n/\ell \rfloor} (A^\ell \setminus \{x\})\right)}{\#A^n}$$

PROOF OF INFINITE MONKEY THEOREM (FORMAL)

$\because$ Let $\ell = |x|$.

$$\Pr\left[x \not\sqsubseteq w \text{ holds for a randomly chosen word } w \in A^n\right]$$

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$$= \frac{\sum_{w' \in A^{(n \bmod \ell)}} \prod_{i \leq \lfloor n/\ell \rfloor} \left(\#A^\ell - 1\right)}{\#A^n}$$

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$$= \frac{\left(\#A^\ell - 1\right)^{\lfloor n/\ell \rfloor} \sum_{w' \in A^{(n \bmod \ell)}} 1}{\#A^n}$$

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$$\leq \Pr\left[x \neq w_i \text{ holds for every decomposed part } w_i \text{ of } w\right]$$

$$= \frac{\#\left(\coprod_{w' \in A^{(n \bmod \ell)}} \prod_{i \leq \lfloor n/\ell \rfloor} \left(A^\ell \setminus \{x\}\right)\right)}{\#A^n}$$

$$= \frac{\left(\#A^\ell - 1\right)^{\lfloor n/\ell \rfloor} \#A^{(n \bmod \ell)}}{(\#A^\ell)^{\lfloor n/\ell \rfloor} \#A^{(n \bmod \ell)}}$$

PROOF OF INFINITE MONKEY THEOREM (FORMAL)

$\because$ Let $\ell = |x|$.

$$\Pr\left[x \not\sqsubseteq w \text{ holds for a randomly chosen word } w \in A^n\right]$$

$$\leq \Pr\left[x \neq w_i \text{ holds for every decomposed part } w_i \text{ of } w\right]$$

$$= \frac{\#\left(\coprod_{w' \in A^{(n \bmod \ell)}} \prod_{i \leq \lfloor n/\ell \rfloor} \left(A^\ell \setminus \{x\}\right)\right)}{\# A^n}$$

$$= \left(\frac{\#A^\ell - 1}{\#A^\ell}\right)^{\lfloor n/\ell \rfloor} = \left(1 - \frac{1}{\#A^\ell}\right)^{\lfloor n/\ell \rfloor} \to 0 \ (n \to \infty)$$

$\therefore$

PROOF OF PARAMETERISED MONKEY THOREM FOR TERMS

$\because$

$$\frac{\#\{[t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi) \mid C_n \not\preceq t\}}{\#\Lambda_n^\alpha(k,\iota,\xi)}$$

$$\leq \frac{\#\{[t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi) \mid C_n \not\preceq u_i \text{ for every } i\}}{\#\Lambda_n^\alpha(k,\iota,\xi)}$$

$$= \frac{\displaystyle\sum_{E \in \mathcal{B}_n^{\lceil \log^{(2)}(n) \rceil}} \# \prod_{i \leq \mathrm{shn}(E)} \left\{ u_i \in U_{E.i}^{\lceil \log^{(2)}(n) \rceil} \mid C_n \not\preceq u_i \right\}}{\#\Lambda_n^\alpha(k,\iota,\xi)}$$

PROOF OF PARAMETERISED MONKEY THOREM FOR TERMS

$\therefore$

$$\frac{\#\{[t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi) \mid C_n \not\leq t\}}{\#\Lambda_n^\alpha(k,\iota,\xi)}$$

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PROOF OF PARAMETERISED MONKEY THOREM FOR TERMS

$$\therefore$$

$$\frac{\#\{[t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi) \mid C_n \not\leq t\}}{\#\Lambda_n^\alpha(k,\iota,\xi)}$$

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PROOF OF PARAMETERISED MONKEY THOREM FOR TERMS

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$\times$

$$= \frac{\displaystyle\sum_{E \in \mathcal{B}_n^{\lceil \log^{(2)}(n) \rceil}} \# \prod_{i \leq \mathrm{shn}(E)} \left\{ u_i \in U_{E.i}^{\lceil \log^{(2)}(n) \rceil} \mid C_n \not\leq u_i \right\}}{\#\Lambda_n^\alpha(k,\iota,\xi)}$$

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

$\because$

$$\frac{\#\{[t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi) \mid C_n \not\preceq t\}}{\#\Lambda_n^\alpha(k,\iota,\xi)}$$

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PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

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Lemma (see our paper for details):

$$\# \prod_{i \leq \mathrm{shn}(E)} \left\{ u_i \in U_{E.i}^{\lceil \log^{(2)}(n) \rceil} \mid C_n \not\preceq u_i \right\}$$

$$\leq \left(\# \prod_{i \leq \mathrm{shn}(E)} U_{E.i}^{\lceil \log^{(2)}(n) \rceil} \right) \left(1 - \frac{1}{c\gamma^{2\lceil \log^{(2)}(n) \rceil}} \right)^{n/4\lceil \log^{(2)}(n) \rceil}$$

$$= \frac{\sum_{E \in \mathcal{B}_n^{\lceil \log^{(2)}(n) \rceil}} \# \prod_{i \leq \mathrm{shn}(E)} \left\{ u_i \in U_{E.i}^{\lceil \log^{(2)}(n) \rceil} \mid C_n \not\preceq u_i \right\}}{\# \Lambda_n^{\alpha}(k, \iota, \xi)}$$

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

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PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

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PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

$$\vdots$$

$$\frac{\#\{[t]_\alpha \in \Lambda^\alpha_n(k,\iota,\xi) \mid C_n \not\preceq t\}}{\#\Lambda^\alpha_n(k,\iota,\xi)}$$

$$\leq \frac{\#\{[t]_\alpha \in \Lambda^\alpha_n(k,\iota,\xi) \mid C_n \not\preceq u_i \text{ for every } i\}}{\#\Lambda^\alpha_n(k,\iota,\xi)}$$

$$= \frac{\sum_{E \in \mathcal{B}_n^{\lceil \log^{(2)}(n) \rceil}} \# \prod_{i \leq \mathrm{shn}(E)} \left\{ u_i \in U_{E.i}^{\lceil \log^{(2)}(n) \rceil} \mid C_n \not\preceq u_i \right\}}{\#\Lambda^\alpha_n(k,\iota,\xi)}$$

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

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$$\# \prod_{i \leq \mathrm{shn}(E)} \left\{ u_i \in U_{E.i}^{\lceil \log^{(2)}(n) \rceil} \mid C_n \not\preceq u_i \right\}$$

$$\leq \left(1 - \frac{1}{c\gamma^{2\lceil \log^{(2)}(n) \rceil}}\right)^{n/4\lceil \log^{(2)}(n) \rceil} \left(\# \prod_{i \leq \mathrm{shn}(E)} U_{E.i}^{\lceil \log^{(2)}(n) \rceil}\right)$$

$$= \frac{\displaystyle\sum_{E \in \mathcal{B}_n^{\lceil \log^{(2)}(n) \rceil}} \# \prod_{i \leq \mathrm{shn}(E)} \left\{ u_i \in U_{E.i}^{\lceil \log^{(2)}(n) \rceil} \mid C_n \not\preceq u_i \right\}}{\# \Lambda_n^{\alpha}(k, \iota, \xi)}$$

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

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Lemma (see our paper for details):

$$\# \prod_{i \leq \mathrm{shn}(E)} \left\{ u_i \in U_{E.i}^{\lceil \log^{(2)}(n) \rceil} \mid C_n \not\leq u_i \right\}$$

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$$= \frac{\displaystyle\sum_{E \in \mathcal{B}_n^{\lceil \log^{(2)}(n) \rceil}} \# \prod_{i \leq \mathrm{shn}(E)} \left\{ u_i \in U_{E.i}^{\lceil \log^{(2)}(n) \rceil} \mid C_n \not\leq u_i \right\}}{\# \Lambda_n^{\alpha}(k, \iota, \xi)}$$

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

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PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

$\therefore$

$$\frac{\#\{[t]_\alpha \in \Lambda^\alpha_n(k,\iota,\xi) \mid C_n \not\preceq t\}}{\#\Lambda^\alpha_n(k,\iota,\xi)}$$

$$\leq \frac{\#\{[t]_\alpha \in \Lambda^\alpha_n(k,\iota,\xi) \mid C_n \not\preceq u_i \text{ for every } i\}}{\#\Lambda^\alpha_n(k,\iota,\xi)}$$

$$= \frac{\sum\limits_{E \in \mathcal{B}_n^{\lceil \log^{(2)}(n) \rceil}} \# \prod\limits_{i \leq \mathrm{shn}(E)} \left\{ u_i \in U_{E.i}^{\lceil \log^{(2)}(n) \rceil} \mid C_n \not\preceq u_i \right\}}{\#\Lambda^\alpha_n(k,\iota,\xi)}$$

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

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$$= \frac{\sum_{E \in \mathcal{B}_n^{\lceil \log^{(2)}(n) \rceil}} \# \prod_{i \leq \mathrm{shn}(E)} \left\{ u_i \in U_{E.i}^{\lceil \log^{(2)}(n) \rceil} \mid C_n \not\leq u_i \right\}}{\#\Lambda^\alpha_n(k,\iota,\xi)}$$

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

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$$\leq \frac{\#\{[t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi) \mid C_n \not\preceq u_i \text{ for every } i\}}{\#\Lambda_n^\alpha(k,\iota,\xi)}$$

$$= \frac{\# \coprod_{E \in \mathcal{B}_n^{\lceil \log^{(2)}(n) \rceil}} \prod_{i \leq \mathrm{shn}(E)} \left\{ u_i \in U_{E.i}^{\lceil \log^{(2)}(n) \rceil} \mid C_n \not\preceq u_i \right\}}{\#\Lambda_n^\alpha(k,\iota,\xi)}$$

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

$\therefore$

Lemma (see our paper for details):

For every $E \in \mathcal{B}_n^{\log^{(2)}(n)}$ *and* $i \leq \mathrm{shn}(E)$,

$$\left\{ u_i \in U_{E.i}^{\lceil \log^{(2)}(n) \rceil} \mid C_n \not\preceq u_i \right\} \leq \# U_{E.i}^{\lceil \log^{(2)}(n) \rceil} - 1$$

$$i.e.\ there\ exists\ u_i \in U_{E.i}^{\lceil \log^{(2)}(n) \rceil}\ such\ that\ C_n \preceq u_i$$

$$= \frac{\# \coprod_{E \in \mathcal{B}_n^{\lceil \log^{(2)}(n) \rceil}} \prod_{i \leq \mathrm{shn}(E)} \left\{ u_i \in U_{E.i}^{\lceil \log^{(2)}(n) \rceil} \mid C_n \not\preceq u_i \right\}}{\# \Lambda_n^{\alpha}(k, \iota, \xi)}$$

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

$\because$

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$$\textit{i.e. there exists } u_i \in U_{E.i}^{\lceil \log^{(2)}(n) \rceil} \textit{ such that } C_n \preceq u_i$$

$$= \frac{\# \coprod_{E \in \mathcal{B}_n^{\lceil \log^{(2)}(n) \rceil}} \prod_{i \leq \mathrm{shn}(E)} \left\{ u_i \in U_{E.i}^{\lceil \log^{(2)}(n) \rceil} \mid C_n \not\preceq u_i \right\}}{\# \Lambda_n^{\alpha}(k, \iota, \xi)}$$

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

$\because$

Lemma (see our paper for details):

For every $E \in \mathcal{B}_n^{\log^{(2)}(n)}$ *and* $i \leq \mathrm{shn}(E)$,

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$$i.e.\ there\ exists\ u_i \in U_{E.i}^{\lceil \log^{(2)}(n) \rceil}\ such\ that\ C_n \preceq u_i$$

$$\leq \frac{\# \coprod_{E \in \mathcal{B}_n^{\lceil \log^{(2)}(n) \rceil}} \prod_{i \leq \mathrm{shn}(E)} \left(\#U_{E.i}^{\lceil \log^{(2)}(n) \rceil} - 1 \right)}{\#\Lambda_n^{\alpha}(k, \iota, \xi)}$$

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

$\because$

Lemma (see our paper for details):

For every $E \in \mathcal{B}_n^{\log^{(2)}(n)}$ *and* $i \leq \mathrm{shn}(E)$,

$$\leq \frac{\# \coprod_{E \in \mathcal{B}_n^{\lceil \log^{(2)}(n) \rceil}} \prod_{i \leq \mathrm{shn}(E)} \left(\# U_{E.i}^{\lceil \log^{(2)}(n) \rceil} - 1 \right)}{\# \Lambda_n^{\alpha}(k, \iota, \xi)}$$

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

$\because$

$$\frac{\#\{[t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi) \mid C_n \not\preceq t\}}{\#\Lambda_n^\alpha(k,\iota,\xi)}$$

$= \Pr[$ $C_n \not\preceq t$ holds for a randomly chosen term t in $\Lambda_n^\alpha(k,\iota,\xi)$ $]$

$\leq \Pr[$ $C_n \not\preceq u_i$ holds for every decomposed part u_i of t $]$

$$= \frac{\displaystyle\sum_{E \in \mathcal{B}_n^{\log^{(2)}(n)}} \#\left(\prod_{i \leq \mathrm{shn}(E)} \{u_i \in U_{E.i}^{\log^{(2)}(n)} \mid C_n \not\preceq u\}\right)}{\displaystyle\sum_{E \in \mathcal{B}_n^{\log^{(2)}(n)}} \#\left(\prod_{i \leq \mathrm{shn}(E)} U_{E.i}^{\log^{(2)}(n)}\right)}$$

PROOF OF PARAMETERISED INFINITE MONKEY THEOREM FOR TERMS

Notation:

$$\overline{\Lambda}_E = \prod_{i \le \mathrm{shn}(E)} \{u \in U_{E.i}^{\log^{(2)}(n)} \mid C_n \not\preceq u\}$$

$$\Lambda_E = \prod_{i \le \mathrm{shn}(E)} U_{E.i}^{\log^{(2)}(n)}$$

$\therefore$ $\#\{[t$ ...

$= \Pr[C_n \not\preceq t$ holds ... $]$

$\le \Pr[C_n \not\preceq u_i$ hold ...

$$= \frac{\sum_{E \in \mathcal{B}_n^{\log^{(2)}(n)}} \#\left(\prod_{i \le \mathrm{shn}(E)} \{u_i \in U_{E.i}^{\log^{(2)}(n)} \mid C_n \not\preceq u\}\right)}{\sum_{E \in \mathcal{B}_n^{\log^{(2)}(n)}} \#\left(\prod_{i \le \mathrm{shn}(E)} U_{E.i}^{\log^{(2)}(n)}\right)}$$

PROOF OF PARAMETERISED INFINITE MONKEY THEOREM FOR TERMS

$\because$ $\#\{[t$

$= \Pr[C_n \not\preceq t$ holds

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Notation:

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$$\Lambda_E = \prod_{i \leq \mathrm{shn}(E)} U_{E.i}^{\log^{(2)}(n)}$$

$$= \frac{\sum_{E \in \mathcal{B}_n^{\log^{(2)}(n)}} \#\overline{\Lambda}_E}{\sum_{E \in \mathcal{B}_n^{\log^{(2)}(n)}} \#\Lambda_E}$$

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

$\because$

$= \mathrm{P}$

$\leq \mathrm{P}$

Lemma (see our paper for details):

There exist constants c and γ such that

$$\#\overline{\Lambda}_E \leq \#\Lambda_E \times \left(1 - \frac{1}{c\gamma^{2c\lceil \log^{(2)}(n) \rceil}}\right)^{n/4c\lceil \log^{(2)}(n) \rceil}$$

holds for every $E \in \mathcal{B}_n^{\log^{(2)}(n)}$.

$$= \frac{\sum_{E \in \mathcal{B}_n^{\log^{(2)}(n)}} \#\overline{\Lambda}_E}{\sum_{E \in \mathcal{B}_n^{\log^{(2)}(n)}} \#\Lambda_E}$$

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

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Lower bound of the # of holes (decomposed parts) of every $E \in \mathcal{B}_n^{\log^{(2)}(n)}$

$$= \frac{\sum_{E \in \mathcal{B}_n^{\log^{(2)}(n)}} \#\overline{\Lambda}_E}{\sum_{E \in \mathcal{B}_n^{\log^{(2)}(n)}} \#\Lambda_E}$$

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

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$$= \frac{\sum_{E \in \mathcal{B}_n^{\log^{(2)}(n)}} \#\overline{\Lambda}_E}{\sum_{E \in \mathcal{B}_n^{\log^{(2)}(n)}} \#\Lambda_E}$$

Asymptotic upper bound of $\#U_{E.i}^{\log^{(2)}(n)}$

$$\left(\#U_{E.i}^{\log^{(2)}(n)} = \mathrm{O}\left(c\gamma^{2c\lceil \log^{(2)}(n) \rceil}\right)\right)$$

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

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holds for every $E \in \mathcal{B}_n^{\log^{(2)}(n)}$.

$$\leq \frac{\sum_{E \in \mathcal{B}_n^{\log^{(2)}(n)}} \#\Lambda_E \times \left(1 - \frac{1}{c\gamma^{2c\lceil \log^{(2)}(n) \rceil}}\right)^{n/4c\lceil \log^{(2)}(n) \rceil}}{\sum_{E \in \mathcal{B}_n^{\log^{(2)}(n)}} \#\Lambda_E}$$

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

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$$\leq \frac{\displaystyle\sum_{E \in \mathcal{B}_n^{\log^{(2)}(n)}} \#\Lambda_E \times \left(1 - \frac{1}{c\gamma^{2c\lceil \log^{(2)}(n)\rceil}}\right)^{n/4c\lceil \log^{(2)}(n)\rceil}}{\displaystyle\sum_{E \in \mathcal{B}_n^{\log^{(2)}(n)}} \#\Lambda_E}$$

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PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

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holds for every $E \in \mathcal{B}_n^{\log^{(2)}(n)}$.

$$\leq \left(1 - \frac{1}{c\gamma^{2c\lceil \log^{(2)}(n) \rceil}}\right)^{n/4c\lceil \log^{(2)}(n) \rceil} \times \frac{\sum_{E \in \mathcal{B}_n^{\log^{(2)}(n)}} \#\Lambda_E}{\sum_{E \in \mathcal{B}_n^{\log^{(2)}(n)}} \#\Lambda_E}$$

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

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holds for every $E \in \mathcal{B}_n^{\log^{(2)}(n)}$.

$$\leq \left(1 - \frac{1}{c\gamma^{2c\lceil \log^{(2)}(n) \rceil}}\right)^{n/4c\lceil \log^{(2)}(n) \rceil}$$

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

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holds for every $E \in \mathcal{B}_n^{\log^{(2)}(n)}$.

$$\leq \left(1 - \frac{1}{c\gamma^{2c\lceil \log^{(2)}(n) \rceil}}\right)^{n/4c\lceil \log^{(2)}(n) \rceil} \to 0 \quad (n \to \infty)$$

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

Lemma (see our paper for details):
There exist constants c and γ such that

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holds for every $E \in \mathcal{B}_n^{\log^{(2)}(n)}$.

$$\leq \left(1 - \frac{1}{c\gamma^{2c\lceil \log^{(2)}(n) \rceil}}\right)^{n/4c\lceil \log^{(2)}(n)\rceil} \to 0 \quad (n \to \infty)$$

This convergence can be proved with elementary analysis (see our paper)

PROOF OF PARAMETERISED INFINITE MONKEY THOREM FOR TERMS

$\because$

$$\frac{\#\{[t]_\alpha \in \Lambda_n^\alpha(k,\iota,\xi) \mid C_n \not\preceq t\}}{\#\Lambda_n^\alpha(k,\iota,\xi)}$$

$= \Pr[\, C_n \not\preceq t$ holds for a randomly chosen term t in $\Lambda_n^\alpha(k,\iota,\xi)\,]$

$\leq \Pr[\, C_n \not\preceq u_i$ holds for every decomposed part u_i of $t\,]$

$$\leq \left(1 - \frac{1}{c\gamma^{2c\lceil \log^{(2)}(n)\rceil}}\right)^{n/4c\lceil \log^{(2)}(n)\rceil} \to 0 \quad (n \to \infty) \qquad \therefore$$

This convergence can be proved with elementary analysis (see our paper)