A sequent calculus for a semi-associative law[1]

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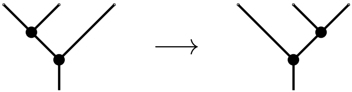
[1] Based on a paper: https://arxiv.org/abs/1803.10080

Introduction

The Tamari order

The partial order on binary trees induced by **right**[2] **rotation**



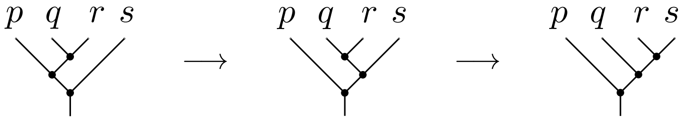
Equivalently, order on bracketings induced by **semi-associativity**

$$(A * B) * C \quad \leq \quad A * (B * C)$$

plus **monotonicity** ($A \leq A'$ and $B \leq B'$ implies $A * B \leq A' * B'$)

[2] Alternatively: left

Example: $(p * (q * r)) * s \leq p * (q * (r * s))$



First studied by Dov Tamari.

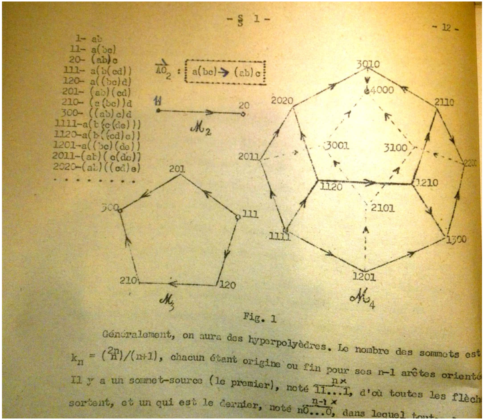


Fig. 1

Généralement, on aura des hyperpolyèdres. Le nombre des sommets est $k_n = \binom{2n}{n}/(n+1)$, chacun étant origine ou fin pour ses n-1 arêtes orienté Il y a un sommet-source (le premier), noté $\overbrace{11\ldots1}^{n\times}$, d'où toutes les flèch sortent, et un qui est le dernier, noté $n\overbrace{0\ldots0}^{n-1\times}$, dans lequel tout

[An excerpt from “Monoïdes préordonnés et chaînes de Malcev,” PhD Thesis, Université de Paris, 1951.]

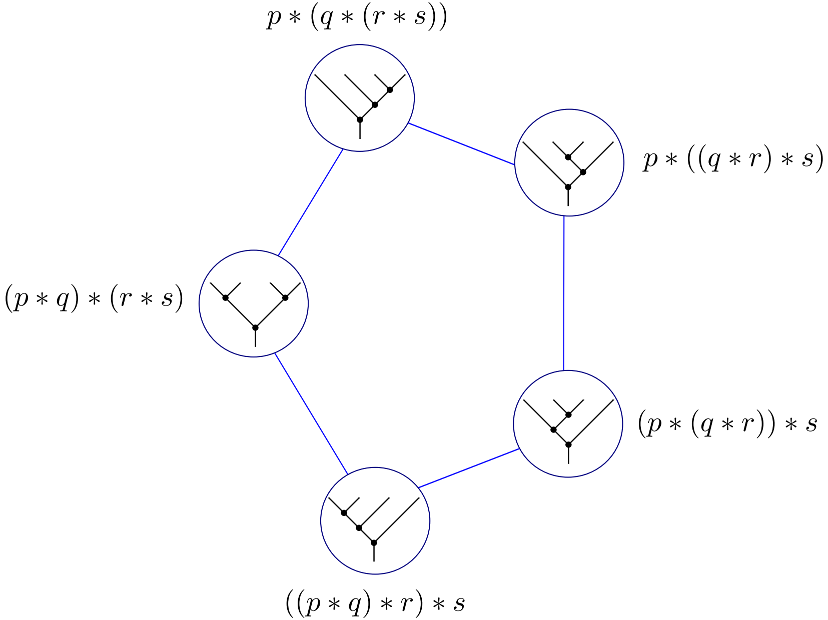
Tamari lattices

Let Y_n be the set of $C_n = \binom{2n}{n}/(n+1)$ Catalan objects of size n, ordered by the Tamari order. Y_n is in fact a **lattice**.

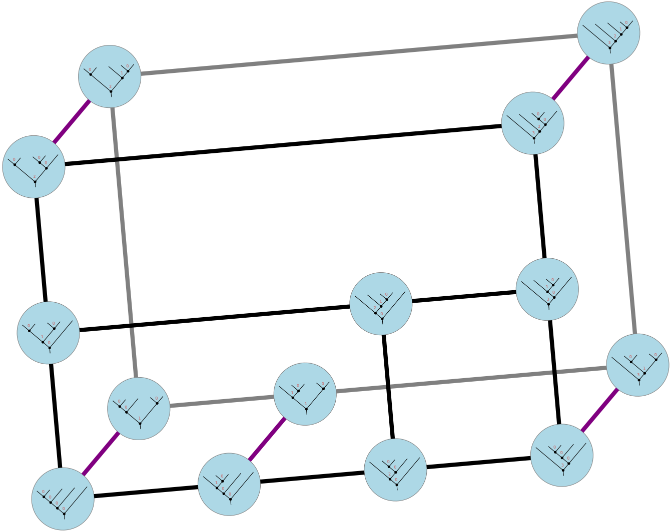
- H. Friedman and D. Tamari, "Problèmes d'associativité: une structure de treillis finis induite par une loi demi-associative," *J. Combinatorial Theory*, vol. 2, 1967.
- S. Huang and D. Tamari, "Problems of associativity: A simple proof for the lattice property of systems ordered by a semi-associative law," *J. Combin. Theory Ser. A*, vol. 13, no. 1, 1972.

The Hasse diagram of Y_n is the skeleton of an $n-1$ dimensional polytope known as an "associahedron".

Y_3



Y_4



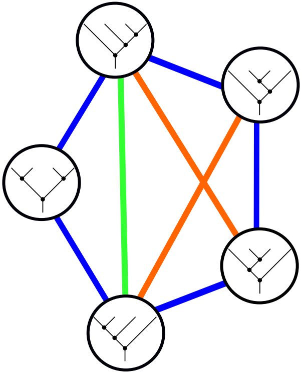
(cf. Knuth, “The associative law, or the anatomy of rotations in binary trees”, https://www.youtube.com/watch?v=Xp7bnx1wDz4)

Counting intervals in Tamari lattices

Theorem (Chapoton 2006)

Let $\mathcal{I}_n = \{\, (A,B) \in \mathsf{Y}_n \times \mathsf{Y}_n \mid A \overset{\text{Tam}}{\leq} B \,\}$. *Then* $|\mathcal{I}_n| = \frac{2(4n+1)!}{(n+1)!(3n+2)!}$.

For example, Y_3 contains 13 intervals:



5 + 5 + 2 + 1 = 13

Note: the formula (A000260) was originally derived by Tutte, but for a completely different family of objects!

Counting rooted maps

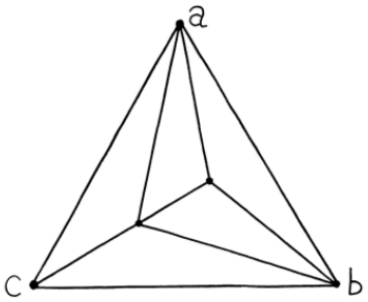


FIGURE 2

We shall prove that

$$(1.5) \qquad \psi_{n,0} = \frac{2}{(n+1)!}(3n+3)(3n+4)\ldots(4n+1)$$

when $n \geqslant 2$. Our main objective in this paper is the complete evaluation of the function $\psi_{n,m}$ (§ 5).

[An excerpt from "A census of planar triangulations," *Canad. J. Math.*, vol. 14, pp. 21–38, 1962]

Counting lambda terms

family of lambda terms	family of rooted maps	OEIS
linear terms	3-valent maps	A062980
planar terms	planar 3-valent maps	A002005
unitless linear	bridgeless 3-valent	A267827
unitless planar	bridgeless planar 3-valent	A000309
normal linear terms/$\sim$	maps	A000698
normal planar terms	planar maps	A000168
normal unitless linear/$\sim$	bridgeless maps	A000699
normal unitless planar	bridgeless planar	A000260

So the Tamari order must be related to lambda calculus?? Perhaps it would be helpful to study it as a logical system...

Sequent calculus

A **formula** is either a product $(A * B)$ or atomic $(p, q, \ldots)$
A **context** $(\Gamma, \Delta, \ldots)$ is a list of formulas
A **sequent** is a pair of a context and a formula $(\Gamma \longrightarrow A)$

A **derivation** is a tree of sequents, constructed via four rules:

$$\frac{}{A \longrightarrow A}\ id \qquad \frac{\Theta \longrightarrow A \quad \Gamma, A, \Delta \longrightarrow B}{\Gamma, \Theta, \Delta \longrightarrow B}\ cut$$

$$\frac{A, B, \Delta \longrightarrow C}{A * B, \Delta \longrightarrow C}\ {*L} \qquad \frac{\Gamma \longrightarrow A \quad \Delta \longrightarrow B}{\Gamma, \Delta \longrightarrow A * B}\ {*R}$$

(Note: no “weakening”, “contraction”, or “exchange” rules.)

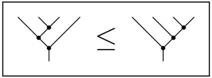
Tamari vs Lambek

These rules are *almost* straight from Lambek[3]...

$$\frac{A, B, \Delta \longrightarrow C}{A * B, \Delta \longrightarrow C} *L \qquad \textit{versus} \qquad \frac{\Gamma, A, B, \Delta \longrightarrow C}{\Gamma, A * B, \Delta \longrightarrow C} *L^{\mathrm{amb}}$$

. . . but this simple restriction makes all the difference!

[3] J. Lambek, "The mathematics of sentence structure," *The American Mathematical Monthly*, vol. 65, no. 3, pp. 154–170, 1958.



Example: $(p * (q * r)) * s \leq p * (q * (r * s))$

$$
\dfrac{
  \dfrac{
    \dfrac{}{p \longrightarrow p}
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{}{q \longrightarrow q}
        \qquad
        \dfrac{\dfrac{}{r \longrightarrow r} \quad \dfrac{}{s \longrightarrow s}}{r, s \longrightarrow r * s}\ R
      }{q, r, s \longrightarrow q * (r * s)}\ R
    }{q * r, s \longrightarrow q * (r * s)}\ L
  }{p, q * r, s \longrightarrow p * (q * (r * s))}\ R
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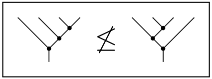
}
$$

$$
\dfrac{\dfrac{p, q * r, s \longrightarrow p * (q * (r * s))}{p * (q * r), s \longrightarrow p * (q * (r * s))}\ L}{(p * (q * r)) * s \longrightarrow p * (q * (r * s))}\ L
$$



Counterexample: $p * (q * (r * s)) \not\leq (p * (q * r)) * s$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\overline{p \longrightarrow p} \quad \cfrac{\overline{q \longrightarrow q} \quad \overline{r \longrightarrow r}}{q, r \longrightarrow q * r}\; R}{p, q, r \longrightarrow p * (q * r)}\; R \qquad \overline{s \longrightarrow s}}{p, q, r, s \longrightarrow (p * (q * r)) * s}\; R}{p, q, r * s \longrightarrow (p * (q * r)) * s}\; L^{\mathrm{amb}}}{p, q * (r * s) \longrightarrow (p * (q * r)) * s}\; L^{\mathrm{amb}}}{p * (q * (r * s)) \longrightarrow (p * (q * r)) * s}\; L$$

Theorem (Completeness)

If $A \overset{\mathrm{Tam}}{\leq} B$ *then* $A \longrightarrow B$.

Theorem (Soundness)

If $\Gamma \longrightarrow B$ *then* $\phi[\Gamma] \overset{\mathrm{Tam}}{\leq} B$*, where*
$\phi[A_0, \ldots, A_n] = ((A_0 * A_1) \cdots) * A_n$ *is the left-associated product*

Proof of completeness (easy)

Reflexivity + transitivity: immediate by *id* and *cut*.

Monotonicity:

$$\dfrac{\dfrac{A \longrightarrow A' \quad B \longrightarrow B'}{A, B \longrightarrow A' * B'}\,R}{A * B \longrightarrow A' * B'}\,L$$

Semi-associativity:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \longrightarrow A} \quad \dfrac{\dfrac{}{B \longrightarrow B} \quad \dfrac{}{C \longrightarrow C}}{B, C \longrightarrow B * C}\,R}{A, B, C \longrightarrow A * (B * C)}\,R}{A * B, C \longrightarrow A * (B * C)}\,L}{(A * B) * C \longrightarrow A * (B * C)}\,L$$

Proof of soundness (mildly satisfying)

Key lemmas about $\phi[-]$:

- "colaxity": $\phi[\Gamma, \Delta] \leq \phi[\Gamma] * \phi[\Delta]$
- $\phi[\Gamma, \Delta] = \phi[\Gamma] \circledast \Delta$, where the (monotonic) right action $- \circledast \Delta$ is defined by $A \circledast (B_1, \ldots, B_n) = ((A * B_1) \cdots) * B_n$

Soundness follows by induction on derivations...

(Case *id*): by reflexivity.

(Case $*L$): $\phi[A * B, \Gamma] = \phi[A, B, \Gamma] \leq C$

(Case $*R$): $\phi[\Gamma, \Delta] \leq \phi[\Gamma] * \phi[\Delta] \leq A * B$

(Case *cut*): $\phi[\Gamma, \Theta, \Delta] = \phi[\Gamma, \Theta] \circledast \Delta \leq (\phi[\Gamma] * \phi[\Theta]) \circledast \Delta \leq (\phi[\Gamma] * A) \circledast \Delta = \phi[\Gamma, A, \Delta] \leq B$

We say that a derivation is **focused** if it only uses $*L$ and the following restricted forms of $*R$ and *id* (and no *cut*):

$$\frac{\Gamma^{\mathrm{irr}} \longrightarrow A \quad \Delta \longrightarrow B}{\Gamma^{\mathrm{irr}}, \Delta \longrightarrow A * B} *R^{\mathrm{foc}} \qquad \frac{}{p \longrightarrow p}\ id^{\mathrm{atm}}$$

where Γ^{irr} denotes an "irreducible" context (atomic on the left)

Theorem (Focusing completeness)

Every derivable sequent has a focused derivation.

Theorem (Coherence)

*Every derivable sequent has **exactly one** focused derivation.*

(Proofs not difficult by standard inductions – no surprises other than that it works!)

Application #1: counting intervals

By the coherence theorem, the problem of counting intervals is equivalent to the problem of counting focused derivations!

Consider the bivariate OGFs $L(z, x)$ and $R(z, x)$, where

$[z^n x^k]L(z, x) = \#$ focused derivations of sequents of the form $\Gamma \longrightarrow A$ with $len(\Gamma) = k$ and $size(A) = n$.

$[z^n x^k]R(z, x) = \#$ focused derivations of sequents of the form $\Gamma^{\mathrm{irr}} \longrightarrow A$ with $len(\Gamma^{\mathrm{irr}}) = k$ and $size(A) = n$.

We have (by the coherence theorem) that

$$|\mathcal{I}_n| = [z^n]L_1(z)$$

where $L_1(z) = [x^1]L(z, x)$.

From the inductive definition of focused derivations...

$$\frac{A, B, \Delta \longrightarrow C}{A * B, \Delta \longrightarrow C} *L \quad \frac{\Gamma^{\mathrm{irr}} \longrightarrow A \quad \Delta \longrightarrow B}{\Gamma^{\mathrm{irr}}, \Delta \longrightarrow A * B} *R^{\mathrm{foc}} \quad \frac{}{p \longrightarrow p} \; id^{\mathrm{atm}}$$

we immediately obtain the following functional equations:

$$\begin{aligned} L(z,x) &= (L(z,x) - xL_1(z))/x + R(z,x) \\ &= x\frac{R(z,x) - R(z,1)}{x-1} \qquad (1) \\ R(z,x) &= zR(z,x)L(z,x) + x \qquad (2) \end{aligned}$$

These can be solved via quadratic method (Cori & Schaeffer '03) to obtain the formula $[z^n]L_1(z) = [z^n]R(z,1) = \frac{2(4n+1)!}{(n+1)!(3n+2)!}$.

Comparison to Chapoton

Chapoton likewise defined a bivariate OGF $\Phi(z,x)$, where x keeps track of "the number of segments along the left border" of the tree at the lower end of the interval, and obtains the following equation:

$$\Phi(z,x) = x^2 z(1 + \Phi(z,x)/x)\left(1 + \frac{\Phi(z,x) - \Phi(z,1)}{x-1}\right) \qquad (3)$$

In fact (3) can be derived from (1) and (2) by taking

$$\Phi(z,x) = R(z,x) - x$$

because Chapoton excludes the case $n = 0$.

(In other words, our proof in the end is very similar to Chapoton's, just a little more systematic.)

Application #2: lattice property

We can use the calculus to give a new proof of the lattice property of the Tamari order, i.e., that each Y_n has joins (and meets).

First step: extend the order to *contexts* via **substitution ordering**, i.e., least relation such that: 1) $\Gamma \longrightarrow A$ implies $\Gamma \leq A$; 2) $\cdot \leq \cdot$; and 3) if $\Gamma_1 \leq \Gamma_2$ and $\Theta_1 \leq \Theta_2$ then $(\Gamma_1, \Theta_1) \leq (\Gamma_2, \Theta_2)$.

Defines a family of posets $\mathsf{F}(\mathsf{Y})_{[n]}$ of **forests with** $n+1$ **leaves**.

Key observation: there is an **adjoint triple**

$$\mathsf{Y}_n \xhookrightarrow{\;i\;} \mathsf{F}(\mathsf{Y})_{[n]} \qquad \phi : \mathsf{F}(\mathsf{Y})_{[n]} \to \mathsf{Y}_n, \quad \psi : \mathsf{Y}_n \to \mathsf{F}(\mathsf{Y})_{[n]}$$

$$\phi[\Gamma] \le A \iff \Gamma \le i[A] \qquad \psi[A] \le \Theta \iff A \le \phi[\Theta]$$

where i is the evident inclusion, ϕ is the left-associated product, and ψ the *maximal decomposition* of a tree along its left border.[4]

We can use this to reduce any join of trees to a join of forests:[5]

$$A \vee B = \phi[\psi[A] \sqcup \psi[B]]$$

[4] The adjunction $\phi \dashv i$ corresponds to soundness & completeness of the sequent calculus, while $\psi \dashv \phi$ follows from focusing completeness.

[5] Since "left adjoints preserve colimits".

Conversely, we can use the fact that $\mathsf{F}(\mathsf{Y})_{[n]}$ lives naturally over the *lattice of compositions ordered by refinement*[6] $\mathrm{O}_{[n]} \cong 2^n$ to reduce any join of forests to a join of compositions, together with joins of trees in Y_m for $m < n$:

$$\Gamma \sqcup \Delta = i[\phi[\Gamma_1] \vee \phi[\Delta_1]], \ldots, i[\phi[\Gamma_k] \vee \phi[\Delta_k]]$$

where the splittings $\Gamma = (\Gamma_1, \ldots, \Gamma_k)$ and $\Delta = (\Delta_1, \ldots, \Delta_k)$ are determined by joining the underlying compositions of Γ and Δ.

[6] In the sense that there is an evident "forgetful mapping" $\mathsf{F}(\mathsf{Y})_{[n]} \to \mathrm{O}_{[n]}$.

Example:

$$A = p * ((q * (r * ((s * t) * u))) * v)$$
$$B = (p * (q * r)) * ((s * t) * (u * v))$$

round	A	$\psi[A]$	B	$\psi[B]$	$\psi[A] \sqcup \psi[B]$
1	$p((q(r((st)u)))v)$	$p, (q(r((st)u)))v$	$(p(qr))((st)(uv))$	$p, qr, (st)(uv)$	$p, A_2 \vee B_2$
2	$(q(r((st)u)))v$	$q, r((st)u), v$	$(qr)((st)(uv))$	$q, r, (st)(uv)$	$q, A_3 \vee B_3$
3	$(r((st)u))v$	$r, (st)u, v$	$r((st)(uv))$	$r, (st)(uv)$	$r, A_4 \vee B_4$
4	$((st)u)v$	s, t, u, v	$(st)(uv)$	s, t, uv	$s, t, A_5 \vee B_5$
5	uv	u, v	uv	u, v	u, v

We conclude that $A \vee B = (p * (q * (r * ((s * t) * (u * v)))))$.

Conclusion

Conclusions and questions

We've given a surprising application of proof theory to combinatorics.

Does the sequent calculus help us to understand any other of the fascinating properties of the associahedra?

What about the permutahedra and other "generalized permutahedra"? (cf. Aguiar and Ardila, arXiv:1709.07504)

What else can we learn by counting proofs?

Selected references

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