Asymptotic Enumeration of Compacted Binary Trees with Height Restrictions

CLA 05/2018

Michael Wallner

joint work with Antoine Genitrini, Bernhard Gittenberger and Manuel Kauers

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May 24th, 2018

Based on the paper:
Asymptotic Enumeration of Compacted Binary Trees,
submitted to a journal.
ArXiv:1703.10031

Creating a compacted tree

Motivation – Efficiently store redundant information

Example

Consider the labeled tree necessary to store the arithmetic expression

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(* (- (* x x) (* y y)) (+ (* x x) (* y y)))
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which represents $(x^2 - y^2)(x^2 + y^2)$.

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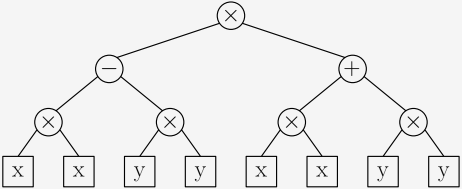

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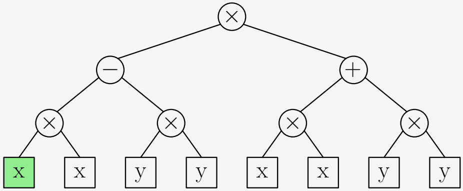

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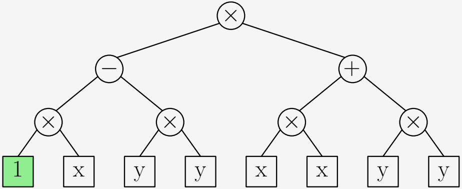

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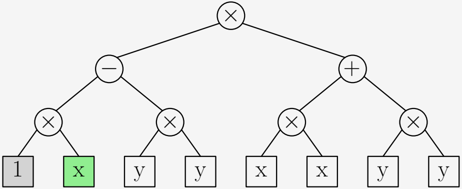

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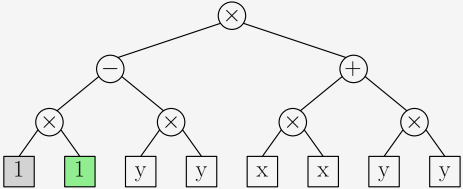

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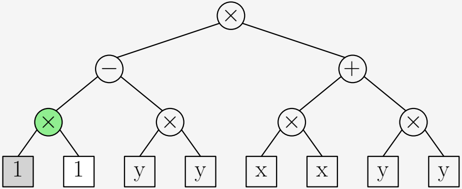

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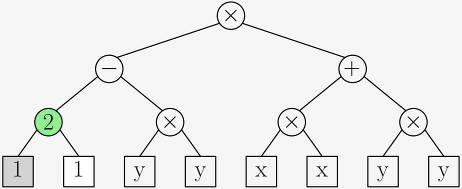

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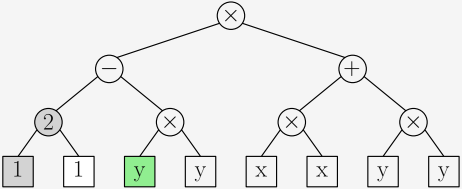

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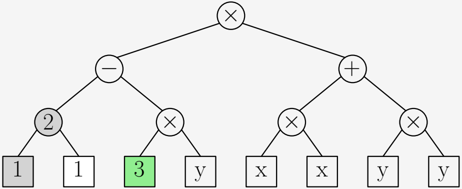

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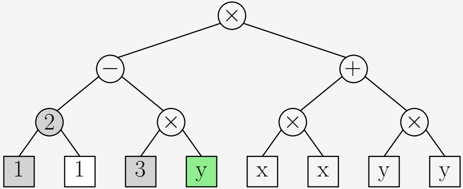

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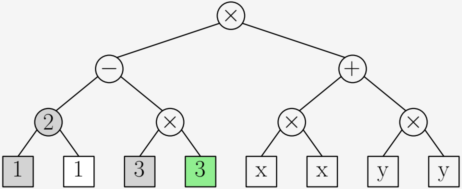

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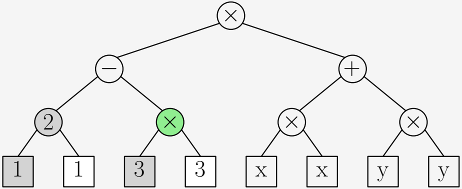

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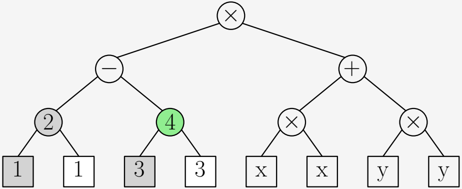

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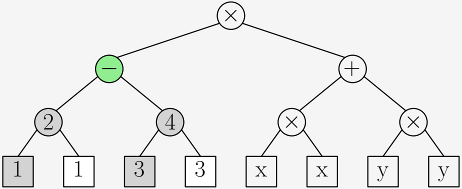

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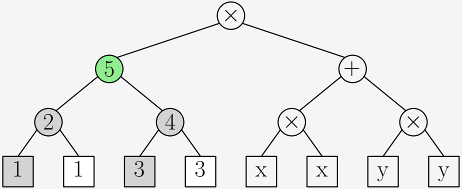

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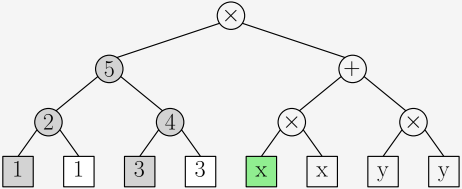

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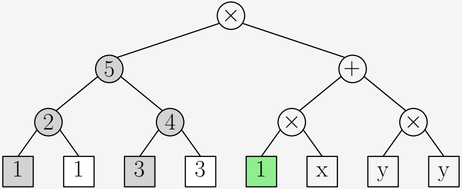

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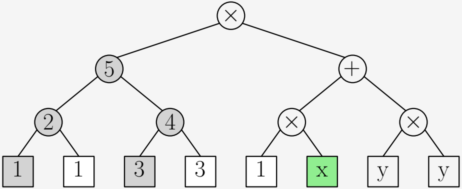

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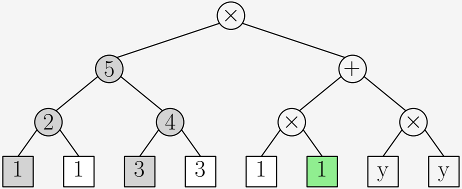

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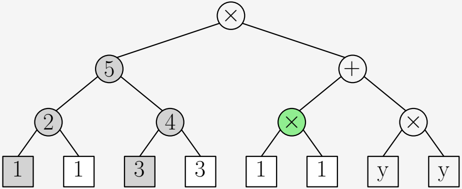

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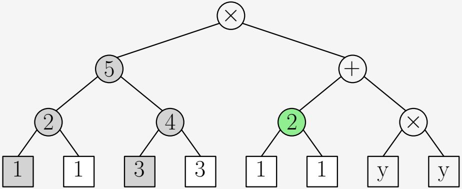

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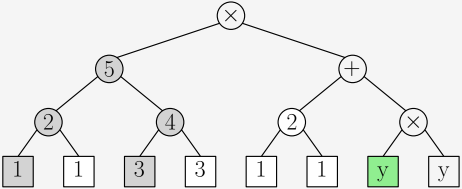

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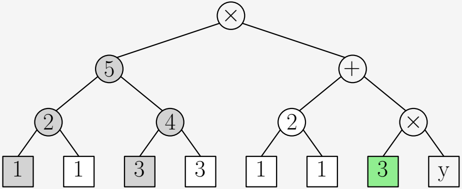

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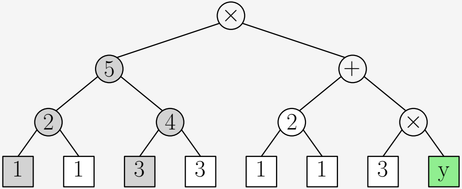

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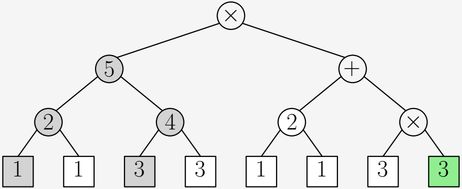

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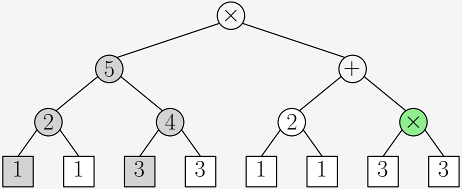

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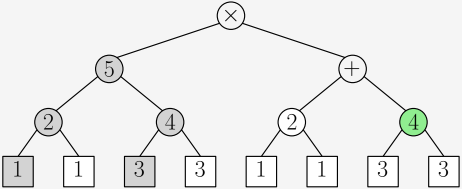

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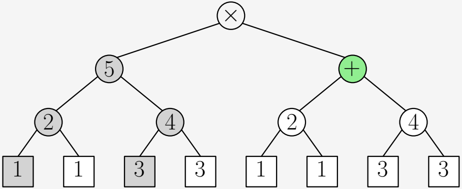

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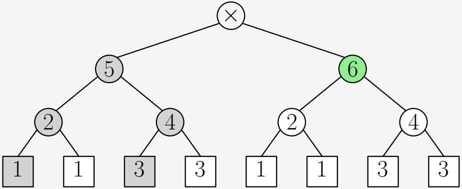

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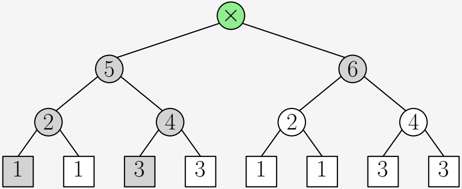

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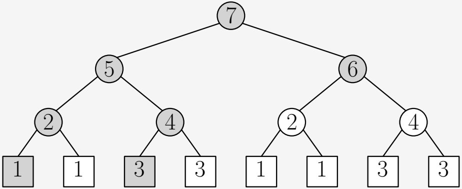

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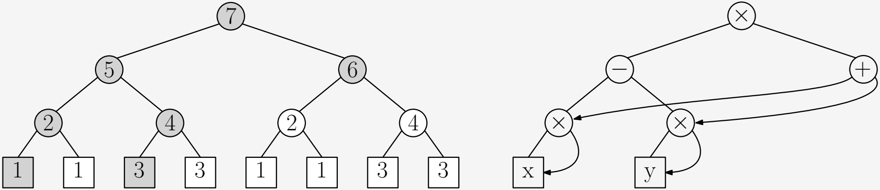

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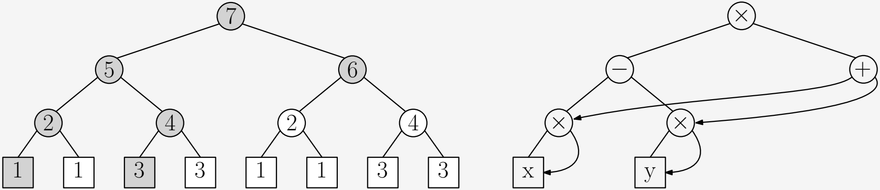

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Definition

Compacted tree is the DAG computed by this procedure.

Compacted trees

- Important property: **Subtrees are unique**
- Efficient algorithm to compute compacted tree
  - Traverse tree post-order
  - If subtree appears twice, delete second one and replace by pointer
    $\rightarrow$ *directed acyclic graph* (DAG)
- Analyzed by [Flajolet, Sipala, Steyaert 1990]: A tree of size n has a compacted form of expected size that is asymptotically equal to

$$C \frac{n}{\sqrt{\log n}},$$

  where C is explicit related to the type of trees and the statistical model.
- Applications: XML-Compression [Bousquet-Mélou, Lohrey, Maneth, Noeth 2015], Compilers [Aho, Sethi, Ullman 1986], LISP [Goto 1974], Data storage [Meinel, Theobald 1998], [Knuth 1968], etc.
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- Important property: **Subtrees are unique**
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  - Traverse tree post-order
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Reverse question

How many compacted trees of (compacted) size n exist?

Compacted trees

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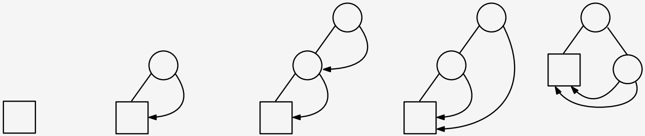

Figure: All compacted binary trees of size $n = 0, 1, 2$.

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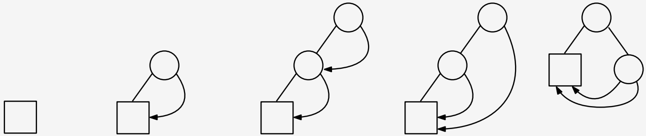

Figure: All compacted binary trees of size $n = 0, 1, 2$.

Example (Compacted binary trees)

size	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
c_n	1	1	3	15	111	1119	14487

$$n! \leq c_n \leq \frac{1}{n+1}\binom{2n}{n} \cdot n!$$

Hence, $c_n = \mathcal{O}(n! 4^n n^{-1/2})$.

Goals of this talk

Goals

1. Understand compacted trees
2. Find a recurrence relation for compacted trees
3. Use exponential generating functions to count DAGs
4. Solve the (simplified) problem(s)

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Methods

1. Recurrence relations
2. Bijections
3. Generating functions
4. Symbolic method
5. Differential equations
6. Singularity analysis
7. Chebyshev polynomials
8. Guess and prove

Building a compacted tree from a binary tree

Idea

Every compacted tree of size n can be build from a binary tree of size n by adding pointers.

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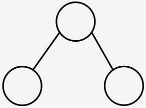

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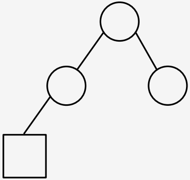

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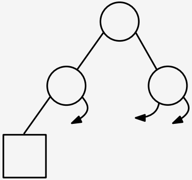

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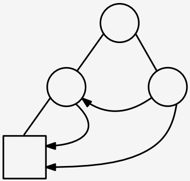

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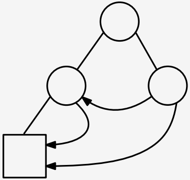

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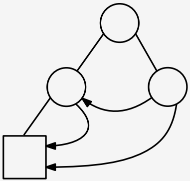

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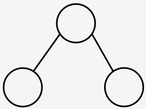

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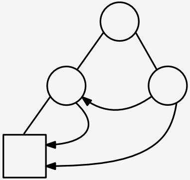

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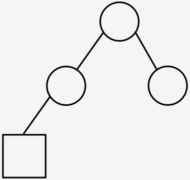

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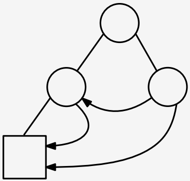

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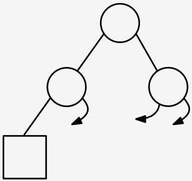

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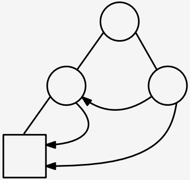

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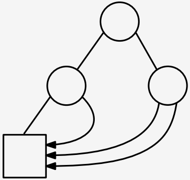

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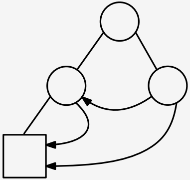

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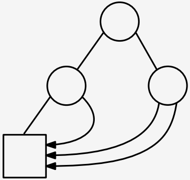

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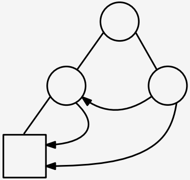

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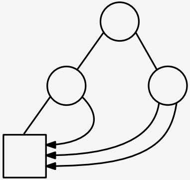

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We call the underlying binary tree from step 1 the *spine*.

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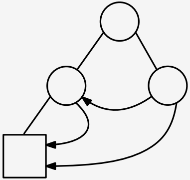

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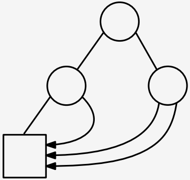

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A bigger example

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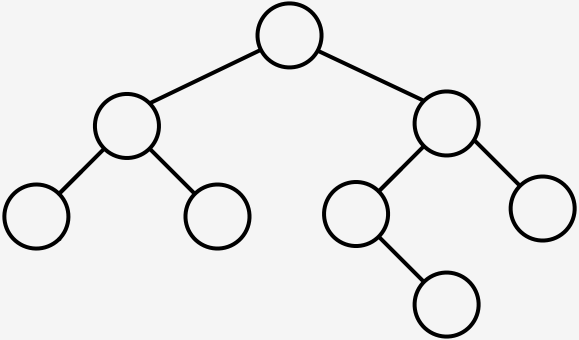

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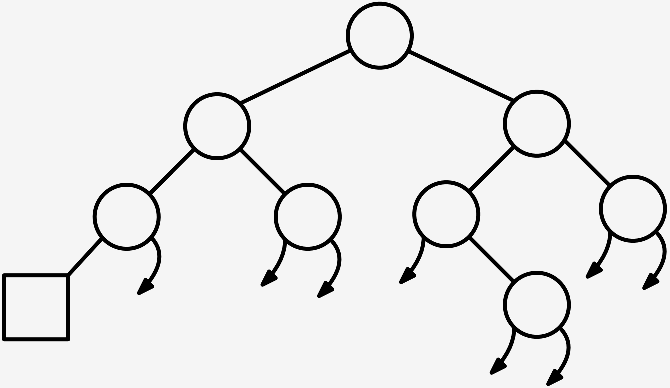

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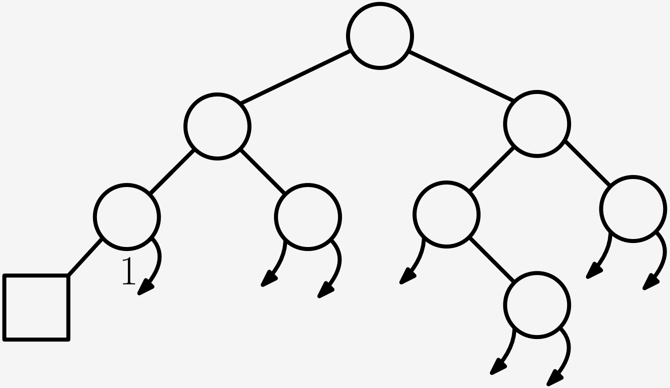

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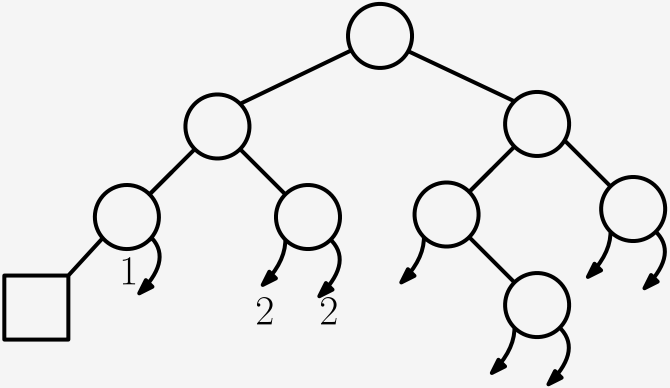

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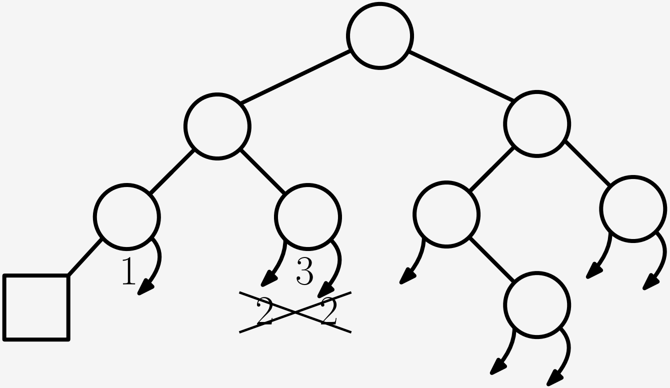

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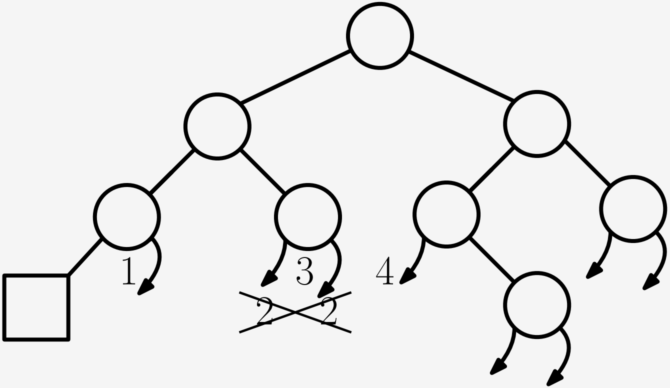

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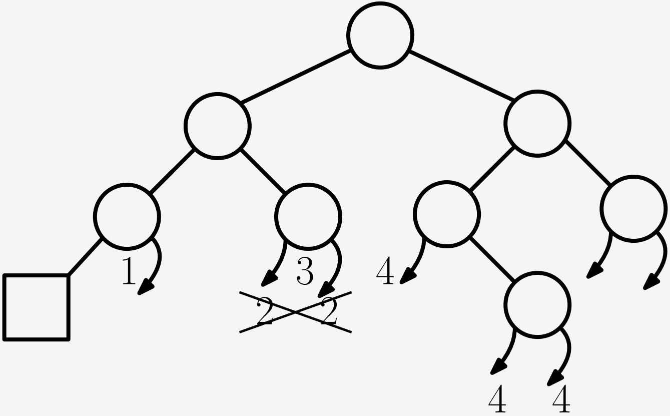

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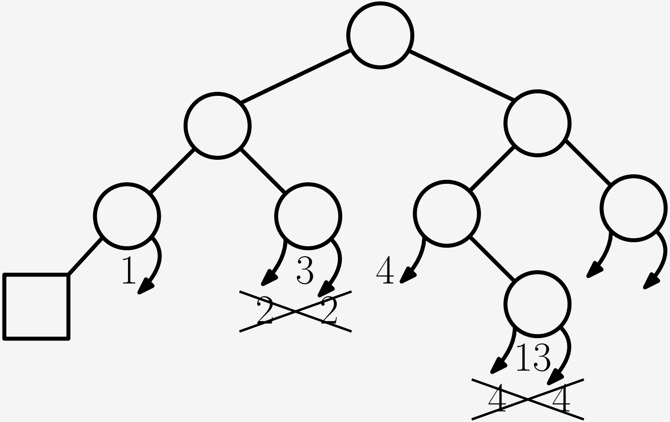

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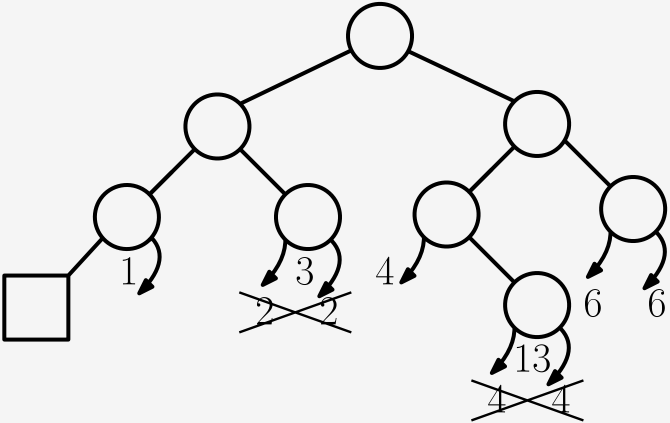

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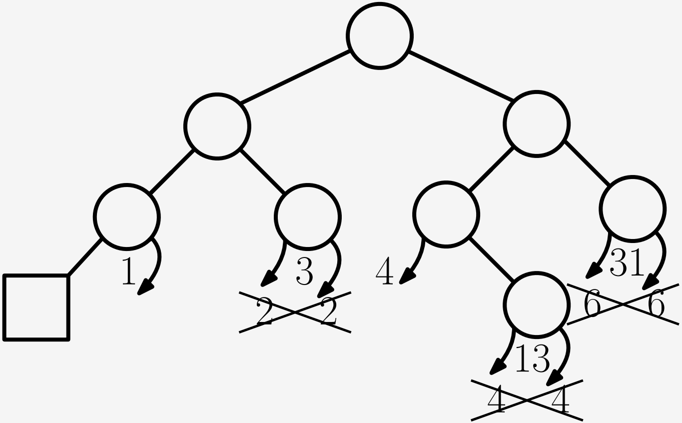

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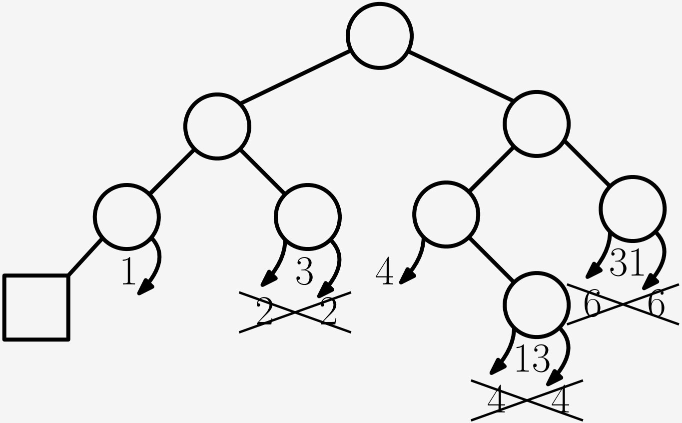

In total, this spine corresponds to $1 \cdot 3 \cdot 4 \cdot 13 \cdot 31 = 4836$ compacted trees.

A recurrence relation

A recurrence for compacted binary trees

Counting formula

Let $n, p \in \mathbb{N}$, then

$$\gamma_{n+1,p} = \sum_{i=0}^{n} \gamma_{i,p}\gamma_{n-i,p+i}, \qquad \text{for } n \geq 1,$$

- Helps us to efficiently compute c_n
- Asymptotic analysis failed (so far)
  One reason: asymptotically every summand matters
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Relaxed compacted binary trees

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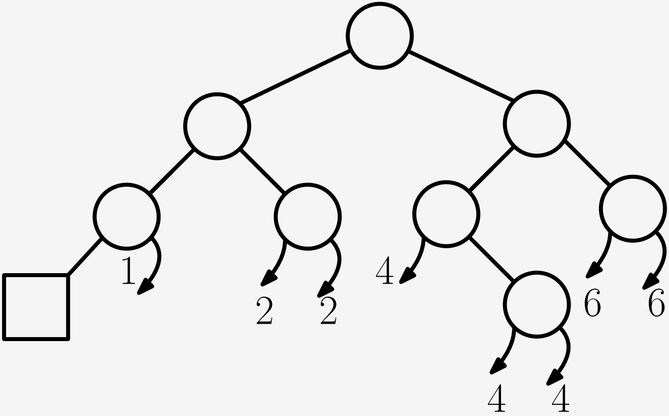

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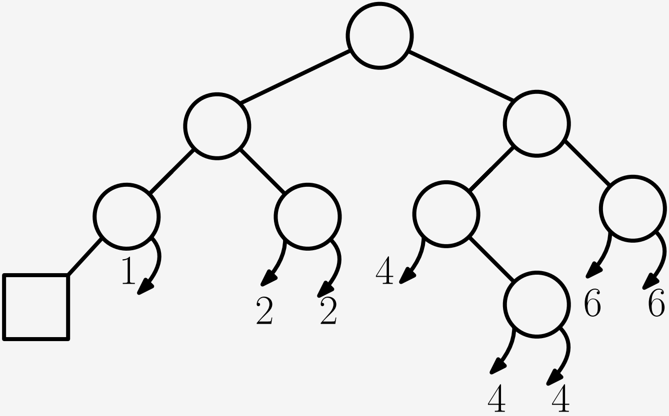
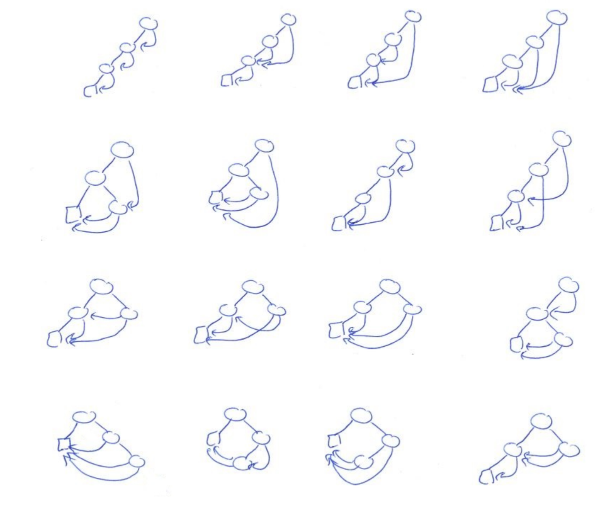


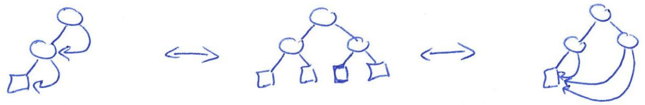
In total, this spine corresponds to $1 \cdot 3 \cdot 4 \cdot 4^2 \cdot 6^2 = 6912$ relaxed trees.
(Recall, that the same spine corresponds to 4836 compacted trees.)

Relaxed compacted binary trees of size 3



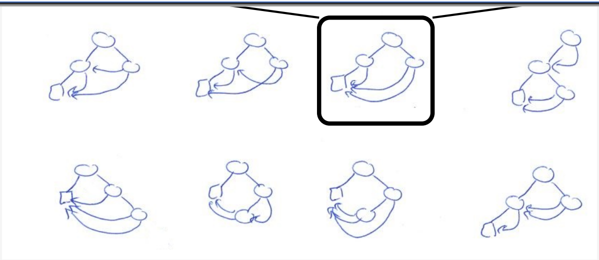
Relaxed compacted binary trees of size 3

The relaxed tree of size 3 which is not a compacted tree



compacted tree   binary tree   relaxed tree

Reason: subtrees not unique



A recurrence for relaxed compacted binary trees

Counting formula

Let $n, p \in \mathbb{N}$, then

$$\delta_{n+1,p} = \sum_{i=0}^{n} \delta_{i,p}\delta_{n-i,p+i}, \qquad \text{for } n \geq 0,$$

$$\delta_{0,p} = p + 1, \qquad \cancel{\delta_{1,p} = p^2 + p + 1}.$$

We are interested in $r_n = \delta_{n,0}$.

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Example (Relaxed binary trees)

size	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
c_n	1	1	3	15	111	1119	14487
r_n	1	1	3	16	127	1363	18628

Operations on trees

Bounded right height

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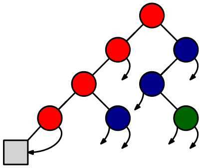

A binary tree with right height 2. Nodes of level 0 are colored in red, nodes of level 1 in blue, and the node of level 3 in green.

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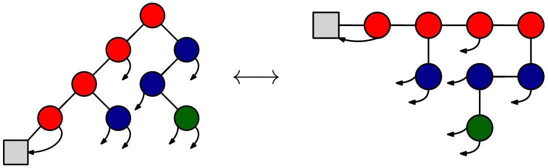

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Figure: Right height ≤ 0.

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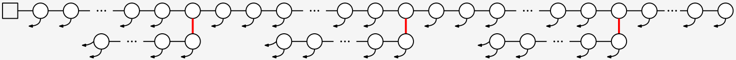
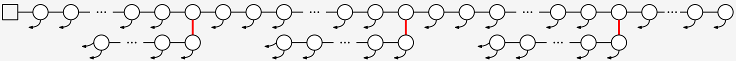


Figure: Right height ≤ 1.

Compacted trees of right height $\leq k$

Figure: Right height ≤ 0.



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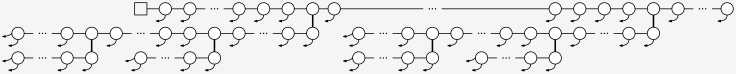
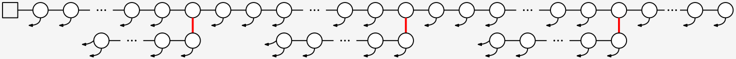


Figure: Right height ≤ 2.

Compacted trees of right height $\leq k$

Figure: Right height ≤ 0.



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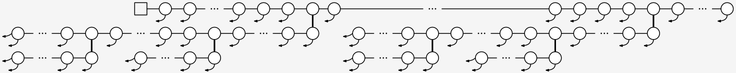

Figure: Right height ≤ 2.

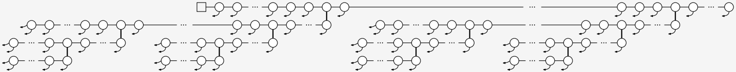

Figure: Right height ≤ 3.

Motivational outlook

Theorem

The number $r_{k,n}$ of relaxed trees with right height at most k is for $n \to \infty$ asymptotically equivalent to

$$r_{k,n} \sim \gamma_k n! \left(4\cos\left(\frac{\pi}{k+3}\right)^2\right)^n n^{-k/2},$$

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Theorem (Main result)

The number $c_{k,n}$ of compacted trees with right height at most k is asymptotically equal to

$$c_{k,n} \sim \kappa_k n! \left(4\cos\left(\frac{\pi}{k+3}\right)^2\right)^n n^{-\frac{k}{2}-\frac{1}{k+3}-\left(\frac{1}{4}-\frac{1}{k+3}\right)\cos\left(\frac{\pi}{k+3}\right)^{-2}},$$

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Main idea: Exponential generating functions

- Asymptotic growth: $n!\rho^n \Rightarrow$ exponential generating functions (EGF)
- Upper bound guarantees positive radius of convergence
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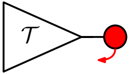
$T(z) \mapsto zT(z)$

Append a new node with a pointer to the class $\mathcal{T}$.



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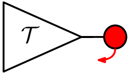

Proof:

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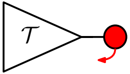

Proof:

$$k![z^k]zT(z) = \underbrace{k}_{k \text{ possible pointers}} \cdot \underbrace{t_{k-1}}_{k-1 \text{ internal nodes}} \qquad \square$$

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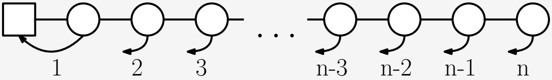

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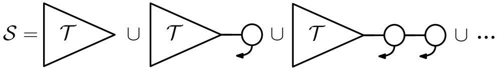
Further constructions

$S : T(z) \mapsto \frac{1}{1-z} T(z)$

Append a (possibly empty) sequence at the root.



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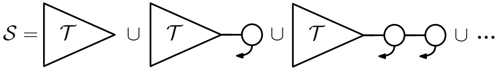
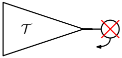


$D : T(z) \mapsto \frac{d}{dz} T(z)$

Delete top node but preserve its pointers.



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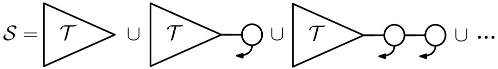

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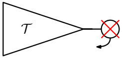
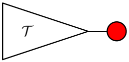


$I : T(z) \mapsto \int T(z)$

Add top node without pointers.



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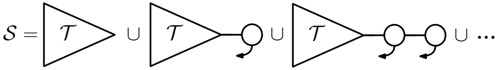

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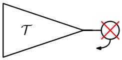

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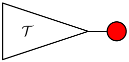
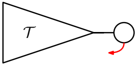


$P : T(z) \mapsto z \frac{d}{dz} T(z)$

Add a new pointer to the top node.



Relaxed binary trees

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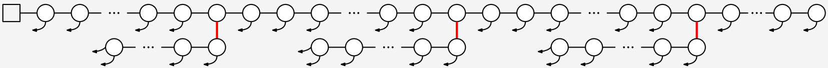

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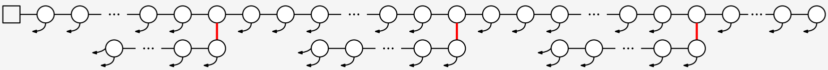

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Decomposition of $R_1(z)$

$$R_1(z) = \sum_{n \geq 0} R_{1,\ell}(z)$$

where $R_{1,\ell}(z)$ is the EGF for relaxed binary trees with exactly ℓ left-subtrees, i.e. ℓ left-edges from level 0 to level 1.

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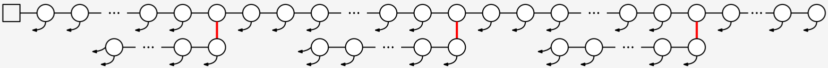

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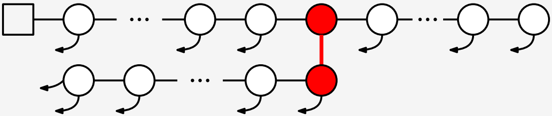
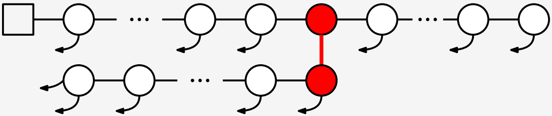
$$R_{1,1}(z) = \ ?$$

Construction of $R_{1,1}(z)$

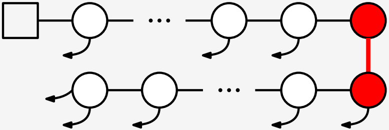


Construction of $R_{1,1}(z)$

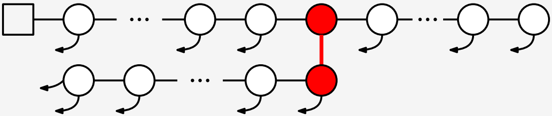


Symbolic specification

1 delete initial sequence

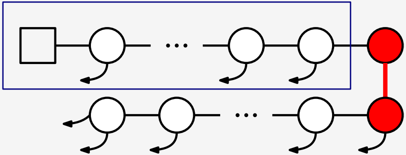


Construction of $R_{1,1}(z)$

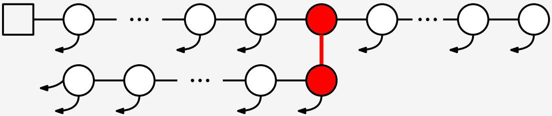


Symbolic specification

1. delete initial sequence
2. decompose



Construction of $R_{1,1}(z)$



Symbolic specification

1. delete initial sequence
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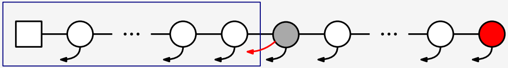

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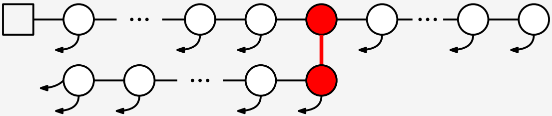

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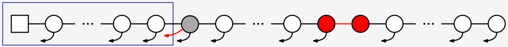
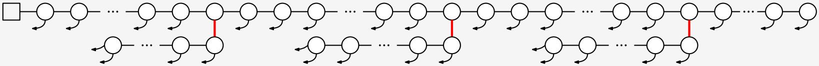


$R_{1,1}(z)$

$$R_{1,1}(z) = \underbrace{S}_{\text{init. seq.}} \circ \underbrace{I}_{\text{lvl 0 node}} \circ \underbrace{S \circ P}_{\text{red pointer and seq.}} \left(\underbrace{zR_{1,0}(z)}_{\text{non empty}} \right)$$

$$R_{1,1}(z) = \frac{1}{1-z} \int \frac{1}{1-z} z \left(zR_{1,0}(z) \right)' \, dz$$

Construction of $R_{1,\ell}(z)$



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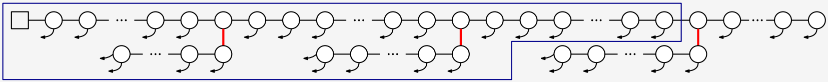
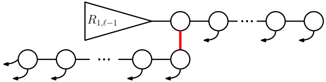


Observation

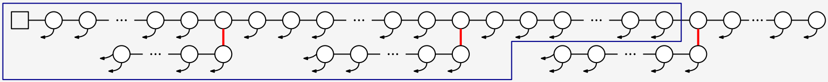
Same structure as for $R_{1,1}(z)$



$$R_{1,\ell}(z) = \frac{1}{1-z} \int \frac{1}{1-z} z \left(z R_{1,\ell-1}(z)\right)' \, dz, \qquad \ell \geq 1,$$

$$R_{1,0}(z) = R_0(z) = \frac{1}{1-z}.$$

Construction of $R_{1,\ell}(z)$



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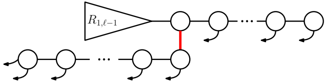

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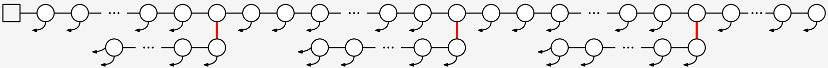
Recall that $R_1(z) = \sum_{\ell \geq 0} R_{1,\ell}(z)$. Summing the previous equation (formally) for $\ell \geq 1$ gives

$$\frac{1-2z}{1-z} R_1'(z) - \frac{1}{1-z} R_1(z) - \left((1-z) R_{1,0}(z)\right)' = 0.$$

Closed form of $R_1(z)$



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We know that $R_{1,0}(z) = \frac{1}{1-z}$ and get

$$(1-2z)\, R_1'(z) - R_1(z) = 0, \qquad \text{with} \qquad R_1(0) = 1.$$

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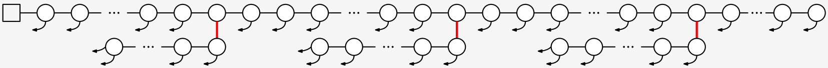

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This directly yields

$$R_1(z) = \frac{1}{\sqrt{1-2z}}.$$

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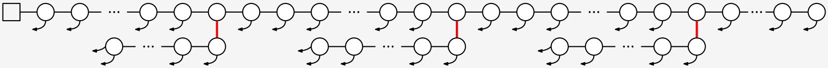

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Therefore we get

$$r_{1,n} = n![z^n]R_1(z) = \frac{n!}{2^n}\binom{2n}{n} = (2n-1)!!.$$

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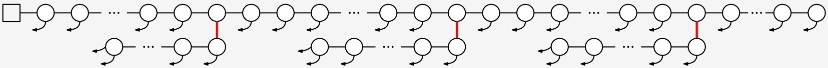

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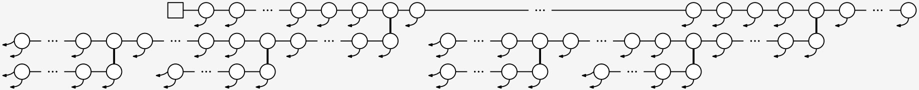
Preprint (ArXiv:1706.07163): [W, 2017, "A bijection of plane increasing trees with relaxed binary trees of right height at most one"].

Bounded right height ≤ 2: $R_2(z)$



Symbolic construction

$$\left(1 - 3z + z^2\right) R_2''(z) + (2z - 3)\, R_2'(z) = 0,$$
$$R_2(0) = 1, \; R_2'(0) = 1,$$

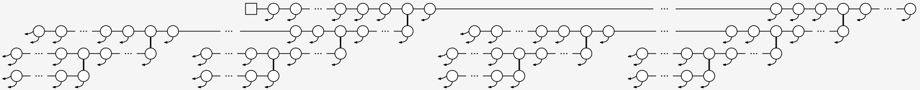
then we get the closed form

$$R_2'(z) = \frac{1}{1 - 3z + z^2},$$

and the coefficients

$$r_{2,n} = n![z^n]R_2(z) = \frac{(n-1)!}{\sqrt{5}} \left(\left(\frac{3+\sqrt{5}}{2} \right)^n - \left(\frac{3-\sqrt{5}}{2} \right)^n \right).$$

Bounded right height ≤ 3: $R_3(z)$



Symbolic construction

$$\left(1-4z+3z^2\right) R_3'''(z) + (9z-6)\, R_3''(z) + 2R_3'(z) = 0,$$

$$R_3(0)=1,\ R_3'(0)=1,\ R_3''(0)=\frac{3}{2},$$

then we get the closed form

$$R_3(z) = \left(\frac{3z-2+\sqrt{3}\sqrt{1-4z+3z^2}}{\sqrt{3}-2}\right)^{1/\sqrt{3}},$$

and the asymptotics of the coefficients

$$r_{3,n} = n![z^n]R_3(z) = \frac{n!}{\sqrt{6}\left(2-\sqrt{3}\right)^{1/\sqrt{3}}} \frac{3^n}{n^{3/2}\sqrt{\pi}} \left(1+\mathcal{O}\left(\frac{1}{n}\right)\right).$$

Differential operators

Theorem

Let $(L_k)_{k\geq 0}$ be a family of differential operators given by

$$L_0 = (1-z),$$
$$L_1 = (1-2z)D - 1,$$
$$L_k = L_{k-1}\cdot D - L_{k-2}\cdot D^2 \cdot z, \qquad k \geq 2.$$

Then the exponential generating function $R_k(z)$ for relaxed trees with right height $\leq k$ satisfies

$$L_k \cdot R_k = 0.$$

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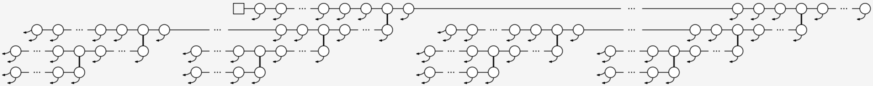
Asymptotics of relaxed trees with bounded right height



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The number $r_{k,n}$ of relaxed trees with right height at most k is for $n \to \infty$ asymptotically equivalent to

$$r_{k,n} \sim \gamma_k n! \left(4\cos\left(\frac{\pi}{k+3}\right)^2\right)^n n^{-k/2},$$

where $\gamma_k \in \mathbb{R} \setminus \{0\}$ is independent of n.

Sketch of proof

1. Let $\ell_{k,i} \in \mathbb{C}[z]$ be such that
$$L_k = \ell_{k,k}(z)D^k + \ell_{k,k-1}(z)D^{k-1} + \ldots + \ell_{k,0}(z).$$
Find recurrences for $\ell_{k,i}(z)$ using Guess'n'Prove techniques.
2. Use singularity analysis directly on differential equation:
3. Exponential growth ρ_k: Roots of coefficient of leading polynomial $\ell_{k,k}(z)$ are candidates.
4. $\ell_{k,k}(z)$ is a transformed Chebyshev polynomial of the second kind. Hence,
$$\rho_k = \frac{1}{4\cos\left(\frac{\pi}{k+3}\right)^2}.$$
5. Subexponential growth: Use the indicial indicial polynomial derived from the $\ell_{k,i}(z)$.
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Only one is singular at ρ_k!
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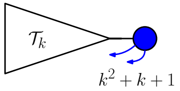
Compacted binary trees

Compacted binary trees

Uniqueness of subtrees



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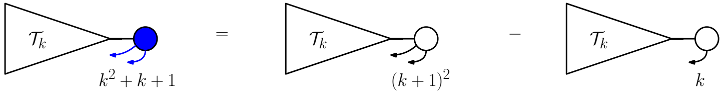

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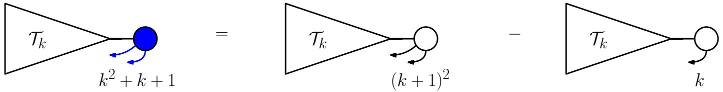

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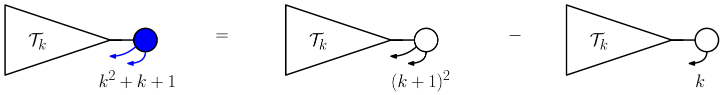

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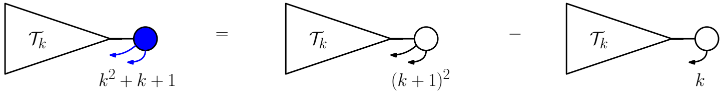

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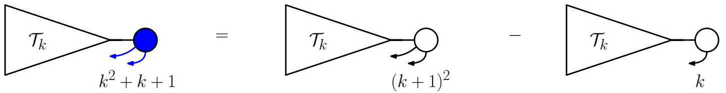

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$$\cdots + (z^2-12z+14)\frac{d^2}{dz^2}C_3(z) + (z-3)\frac{d}{dz}C_3(z) = 0.$$

Asymptotics of compacted trees with bounded right height

Theorem (Main result)

The number $c_{k,n}$ of compacted trees with right height at most k is asymptotically equal to

$$c_{k,n} \sim \kappa_k \, n! \left(4 \cos\left(\frac{\pi}{k+3}\right)^2\right)^n n^{-\frac{k}{2} - \frac{1}{k+3} - \left(\frac{1}{4} - \frac{1}{k+3}\right) \cos\left(\frac{\pi}{k+3}\right)^{-2}},$$

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- We derived a **symbolic method** on exponential generating functions,
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Relaxed trees with right height at most k

$$r_{k,n} \sim \gamma_k n! \left(4\cos\left(\frac{\pi}{k+3}\right)^2\right)^n n^{-k/2}$$

Comparing compacted and relaxed trees

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Corollary (Proportion of compacted among relaxed trees)

$$\frac{c_{k,n}}{r_{k,n}}$$

Comparing compacted and relaxed trees

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Next steps

- Enumeration of compacted trees *without* height restrictions
- Different tree structures, like e.g. ternary trees
- Analyze shape parameters, like height, width, profile, ...

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