# Some parameters in linear lambda-terms

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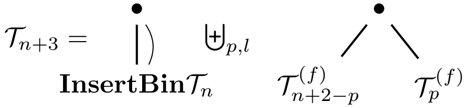
# Table of contents

1. Number of free variables
   - Specification
   - Correspondence between maps and linear lambda-terms
   - Asymptotic distribution
2. Specifications with various parameters
   - Specifications
   - Asymptotic distribution
3. Some problems to address
4. References

# Counting plain and closed terms

A **plain linear-lambda term** can either be an application between two terms, or a smaller term where a binary node is inserted on the top left or on the top right of an existing node, an abstraction is added above the root and connected to the new leaf:

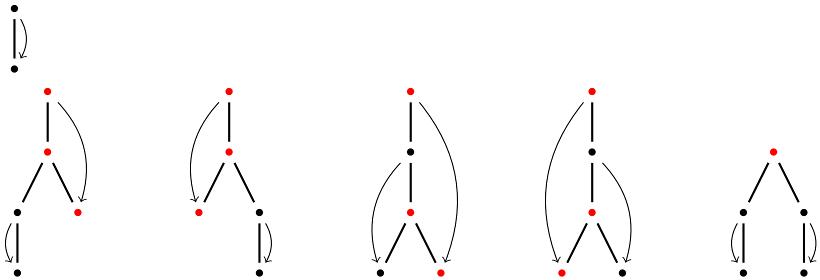



$$f(z) = z + z^2 + zf^2(z) + 2z^4\partial_z f(z).$$

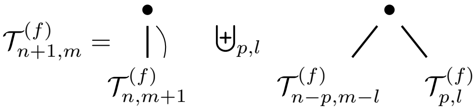
**Closed terms** obey the same specification apart from the initial terms:

$$f(z) = z^2 + zf^2(z) + 2z^4\partial_z f(z).$$

Construction of closed terms of size 5 from the term of size 2.



We can also give a specification where we take into account the number of free variables:




$$f(z,u) = uz + zf^2(z,u) + 2\partial_u f(z,u)$$

**Some coefficients of the enumerating array.**

| n \ m | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 1 | 0 | **1** | 0 | 0 | 0 |
| 2 | **1** | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | **1** | 0 | 0 |
| 4 | 0 | **4** | 0 | 0 | 0 |
| 5 | **5** | 0 | 0 | **2** | 0 |
| 6 | 0 | 0 | **16** | 0 | 0 |
| 7 | 0 | **50** | 0 | 0 | **5** |
| 8 | **60** | 0 | 0 | **64** | 0 |
| 9 | 0 | 0 | **350** | 0 | 0 |
| 10 | 0 | **960** | 0 | 0 | **256** |
| 11 | **1105** | 0 | 0 | **2100** | 0 |

# Correspondence with trivalent maps

## Definition

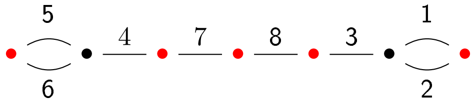
Up to a labelling, a **rooted trivalent map** of size $n = 6k + 2$ is a root in $[\![1, n]\!]$ and couple of two permutations over $[\![1, n]\!]$, such that the first one, denoted by $\sigma_{\color{red}\bullet}$, is an involution with no fixed point, the second one, denoted by $\sigma_\bullet$, is of order $3$ and has two fixed points $a$ and $b$ (including the root), and such that $\sigma_{\color{red}\bullet}$, $\sigma_\bullet$ and $(a\ b)$ generate a transitive group.

**Remark**

We can represent rooted trivalent maps with graphs with degree $2$ nodes $\color{red}\bullet$ and degree $3$ nodes $\bullet$ where $\sigma_{\color{red}\bullet}$, $\sigma_{\bullet}$ act on the edges.

**Example**

$\rho = 7$, $\sigma_{\color{red}\bullet} = (12)(38)(56)(74)$, $\sigma_{\bullet} = (123)(456)$.



**Theorem (O.Bodini, D. Gardy, A. Jacquot, 2013)**

$\mathcal{T}^{(f)}_{3k+2,0}$ is in bijection with trivalent rooted map. In particular the generating function of closed terms is given by:

$$[z^{3k+2}, u^0]\widetilde{f} = [z^{6k+2}]z^3\partial_z \log(e^{z^3/3} \odot e^{z^2/2})$$

where $\odot$ is the exponential Hadamard product in $z$.

**Theorem (Consequence of the correspondence of N. Zeilberger, 2017)**

We can generalize this correspondence to plain terms. In particular, the generating function of $\mathcal{T}^{(f)}$ is given by:

$$[z^{3k+c(m)}, u^m]\widetilde{f} = [z^{6k+\widetilde{c}(m)}, u^m] z^3 \partial_z \log(e^{z^3/3+uz} \odot e^{z^2/2})$$

where $c(m) = 2$ if $m = 0\ mod[3]$, $1$ if $m = 1\ mod[3]$, $0$ if $m = 2\ mod[3]$
$\widetilde{c}(m) = 2$ if $m = 0\ mod[3]$, $6$ if $m = 1\ mod[3]$, $4$ if $m = 2\ mod[3]$

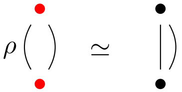
**Remark**

Thanks to a theorem of E. A. Bender, we can prove the subsequent proposition using the coefficients of the following function:

$$e^{z^3/3+uz} \odot e^{z^2/2}$$

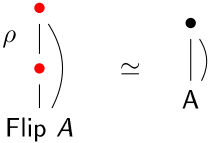
# Correspondence between maps and closed terms

- Initial term:



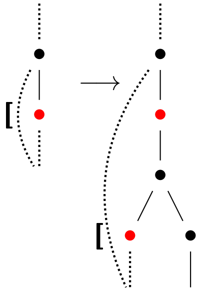

- Application (Flip changes root node into a black node with no root edge):

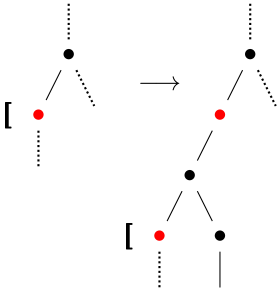

- Add an abstraction above the root (if A is not closed):



• Insertion of a binary node on the top right:
([ marks the node above which we make the insertion)

Above the son of an abstraction | Above the son of an application

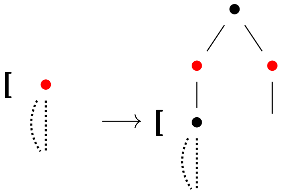




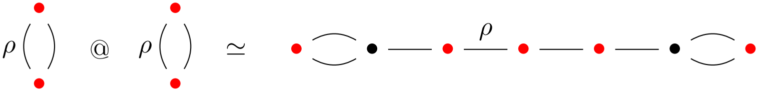
**Remark**

We have to change the added leaf into a red one when we connect it to an abstraction.

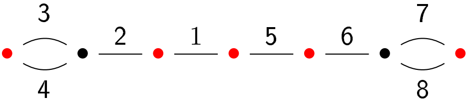
Above the root node



# Example: $(\lambda x \cdot x)(\lambda x \cdot x)$




In terms of permutations, up to a relabelling of the edges we will have:

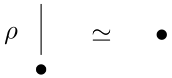



$$\sigma_\bullet = (234)(678)$$
$$\sigma_{\color{red}\bullet} = (12)(34)(56)(78)$$

# Generalisation to plain terms

• In plain terms free leaves are black contrary to closed leaves, they are also followed by an edge.

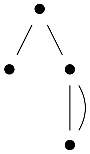
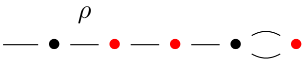
• We add the following element:



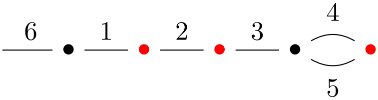

• Application: to make a product with a term containing the of term size $1$, we connect it directly to the root node of the application.

• Connecting an abstraction: to connect an abstraction to a black leaf of a term, if it is the term of size $1$, we just transform it into the term of size $2$, else we transform the leaf into a red one before the connection.

# Example: $x\ \lambda y \cdot y$

$x\lambda y \cdot y \quad \simeq$  $\simeq$ 


Up to a relabelling of the edges, we can give the following representation:




$$\sigma_{\bullet} = (345)$$
$$\sigma_{\color{red}\bullet} = (23)(45)$$

# Average value and variance

**Proposition**

The expectation and the variance of the number of free variables in random terms of size $n$ have the following asymptotic:

$$\mathbb{E}X_n \sim \mathbb{V}X_n \sim (2n)^{1/3}$$

It follows directly from the specifications of plain terms and of terms with a given number of free variables.

# Asymptotic distribution

**Proposition**

$$\frac{X_n - (2n)^{1/3}}{(2n)^{1/6}} \xrightarrow{law} \mathcal{N}(0,1)$$

We compute the coefficients $\phi_n(m) = [z^n, u^m]e^{z^3/3+uz} \odot e^{z^2/2}$. Then we show that asymptotically the discrete derivative $\phi_n(m+3) - \phi_n(m)$ is $\left((2n)^{1/3} - m\right)/(2n)^{1/3}$ times $\phi_n(m)$.

# Specifications with various parameters

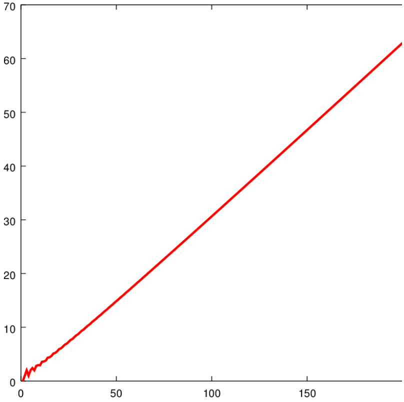
| Parameters | Equation(closed terms) | Equation(plain terms) |
|---|---|---|
| Size only | $f = z^2 + zf^2 + 2z^4\partial_z f$ | $f = z + z^2 + zf^2 + 2z^4\partial_z f$ |
| Abstractions | $f = uz^2 + zf^2 + 2uz^4\partial_z f$ | $f = z + uz^2 + zf^2 + 2uz^4\partial_z f$ |
| Free variables | / | $f = uz + zf^2 + z\partial_u f$ |
| Head abstractions | $f = uz^2 + zf^2_{\vert u=1} + 2uz^4\partial_z f$ | $f = z + uz^2 + zf^2_{\vert u=1} + 2uz^4\partial_z f$ |

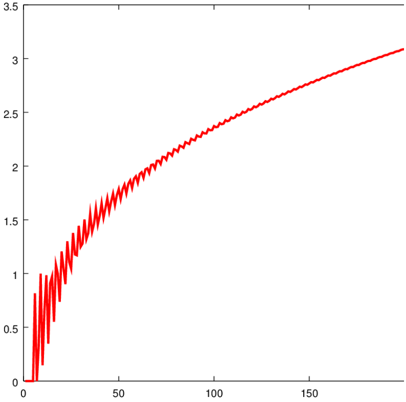
**Remarks**

-Closed lambda-terms of size $3k+2$ have exactly $k+1$ free variables because each time we add a binary node we have to add an abstraction, so we can simply write a specification with two parameters.

-As the number of free variables is small compared to the total number of variables, on average, the number of abstractions in plain terms of size $n$ is concentrated near $n/3$.

# Expectation and variance of the number of abstractions



# Specifications involving redexes

- size, ♯abstractions, ♯redexes in closed terms
$f = uz^2 + zf^2 + 2(v-1)uz^4 f\partial_z f + 2uz^4\partial_z f + (v-1)u^2z^3\partial_u f$

- size, ♯abstractions, ♯redexes in plain terms
$f = z + uz^2 + zf^2 + 2(v-1)uz^4 f\partial_z f + 2uz^4\partial_z f + (v-1)u^2z^3\partial_u f$

# Distribution of redexes in closed terms

$\rightarrow$ if we only consider the terms of the form
$\lambda x \cdot (\textbf{insertbin} T_{n-3})$ or $T_{n-3}(\lambda x \cdot x)$ or $(\lambda x \cdot x)T_{n-3}$,
we can approximate the number of redexes in closed terms of size $n = 3k + 2$ with the following process:

$$R_n = R_{n-3} + B_n$$

where $R_2 = 0$ and $\{B_n\}$ are Bernoulli variables with parameter
$\mathbb{P}(B_n = 1) = \frac{1 - T_n/T_{n-3}}{6} + \frac{T_n/T_{n-3}}{2} \sim \frac{1 - e/2n}{6} + \frac{e}{4n}$.

# Some problems to address

- Translate parameters in lambda-terms into parameters in maps.
- Give a systematic method to solve the differential equation corresponding to the parameter of interest.
- Extend this work to affine terms.
- Work with non uniform distributions of terms.

# References


- O. Bodini, D. Gardy, A. Jacquot. "Asymptotics and random sampling for BCI and BCK lambda terms"
- N. Zeilberger. "Linear lambda terms as invariants of rooted trivalent maps"
- S. A. Vidal. "An optimal algorithm to generate rooted trivalent diagrams and rooted triangular map"
- O. Bodini, S. Dovgal, J. Courtiel, H-K. Hwang. "Asymptotic distribution of parameters in random maps"