Counting Environments and Closures

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  - For instance $\langle \lambda \underline{0}, \Box \rangle : \langle \lambda\lambda \underline{0}, \Box \rangle : \Box$ is an environment :

$$\begin{array}{ccc} \underline{0} & \mapsto & \langle \lambda \underline{0}, \Box \rangle \\ \underline{1} & \mapsto & \langle \lambda\lambda \underline{0}, \Box \rangle \\ \underline{2} & \mapsto & \textbf{nothing} \end{array}$$

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  - For instance $\langle \underline{1}\, \underline{0}, \quad \langle \lambda \underline{0}, \square \rangle : \langle \lambda\lambda \underline{0}, \square \rangle : \square \rangle$ is a closure.
    - ★ If one *"reduce"* this closure, one gets $\lambda\lambda\underline{0}$.

The equations for closures

The definition of the set of closures and of the set of environments is recursive

$$
\begin{aligned}
\mathcal{C}los &::= \langle \Lambda, \mathcal{E}nv \rangle \\
\mathcal{E}nv &::= \Box \mid \mathcal{C}los : \mathcal{E}nv
\end{aligned}
$$

Counting environments and closures

- λ-terms:

$$
\begin{aligned}
|0| &= 1 \\
|\mathsf{S}\ n| &= |\underline{n}| = |n| + 1
\end{aligned}
\qquad
\begin{aligned}
|M\,N| &= |M| + |N| + 1 \\
|\lambda M| &= |M| + 1\,.
\end{aligned}
$$

Counting environments and closures

- λ-terms:

$$\begin{array}{lllll} |0| & = & 1 & & \\ |\mathsf{S}\ n| & = & |\underline{n}| & = & |n| + 1 \end{array} \qquad \begin{array}{lll} |M\,N| & = & |M| + |N| + 1 \\ |\lambda M| & = & |M| + 1\,. \end{array}$$

- Closures and environments:

$$|\langle M, e\rangle| = |M| + |e| \qquad |\mathfrak{c} : e| = |\mathfrak{c}| + |e| \qquad |\Box| = 0\,.$$

The first environments and closures

Here are the first plain environments:

size	environments	total
0	$\square$	1
1	$\langle \underline{0}, \square \rangle : \square$	1
2	$\langle \underline{0}, \square \rangle : \langle \underline{0}, \square \rangle : \square$ $\langle \underline{0}, \langle \underline{0}, \square \rangle : \square \rangle : \square$ $\langle \lambda \underline{0}, \square \rangle : \square, \quad \langle \underline{1}, \square \rangle : \square$	4

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and plain closures:

size	closures	total
0		0
1	$\langle \underline{0}, \square \rangle$	1
2	$\langle \underline{0}, \langle \underline{0}, \square \rangle \rangle$ $\langle \lambda \underline{0}, \square \rangle \quad \langle \underline{1}, \square \rangle$	3

▸ Closed Closures

Equations for plain closures and environments

- Set equations

$$\begin{aligned}\mathcal{E}_\infty &= \mathcal{C}_\infty : \mathcal{E}_\infty \mid \square \\ \mathcal{C}_\infty &= \langle \mathcal{L}_\infty, \mathcal{E}_\infty \rangle\,.\end{aligned}$$

Equations for plain closures and environments

- Set equations

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- Equations for generating functions

$$\begin{array}{rcl} E_\infty(z) & = & C_\infty(z) E_\infty(z) + 1 \\ C_\infty(z) & = & L_\infty(z)\, E_\infty(z). \end{array}$$

Generating functions

- Generating functions

$$
\begin{aligned}
E_\infty(z) &= \frac{1 - \sqrt{1 - 4L_\infty(z)}}{2L_\infty(z)} \\
C_\infty(z) &= \frac{1}{2}\left(1 - \sqrt{1 - 4L_\infty(z)}\right) .
\end{aligned}
$$

Two sources of singularities:

- Generating functions

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- The dominant singularity ρ_∞ of $L_\infty(z)$

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Two sources of singularities:

  - The dominant singularity ρ_∞ of $L_\infty(z)$
  - The root ρ of $1-4\,L_\infty(z)$ $\rho < \rho_\infty$

- Radius of convergence in 0

$$\rho = \frac{1}{10}\left(25-\sqrt{545}\right) \doteq 0.165476\,.$$

- Newton-Puiseux expansions

$$E_\infty(z) = 2 + -\frac{1}{4}\sqrt{\frac{5}{47}\left(109 + 35\sqrt{545}\right)}\sqrt{1 - \frac{z}{\rho}} + O\left(\left|1 - \frac{z}{\rho}\right|\right)$$

$$C_\infty(z) = \frac{1}{2} + \frac{2\sqrt{\frac{10\left(48069\sqrt{5} - 10295\sqrt{109}\right)}{65\sqrt{109} - 301\sqrt{5}}}}{3\sqrt{545} - 77}\sqrt{1 - \frac{z}{\rho}} + O\left(\left|1 - \frac{z}{\rho}\right|\right)$$

Asymptotics

The numbers e_n and c_n of plain environments and closures of size n, respectively, admit the following asymptotic approximations:

$$e_n \sim C_e \cdot \rho^{-n} n^{-3/2}$$
$$c_n \sim C_c \cdot \rho^{-n} n^{-3/2}$$

where

$$C_e = \frac{\sqrt{\frac{5}{47}\left(109 + 35\sqrt{545}\right)}}{8\sqrt{\pi}} \doteq 0.699997,$$

$$C_c = \frac{\sqrt{\frac{10\left(48069\sqrt{5} - 10295\sqrt{109}\right)}{65\sqrt{109} - 301\sqrt{5}}}}{\sqrt{\pi}\left(77 - 3\sqrt{545}\right)} \doteq 0.174999$$

and

$$\rho = \frac{1}{10}\left(25 - \sqrt{545}\right) \doteq 0.165476 \quad \text{giving} \quad \rho^{-n} \doteq 6.04315^n.$$

Holonomicity

- The generating functions are algebraic

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- then we get holonomic functions

Holonomicity

- The generating functions are algebraic
- then we get holonomic functions
- then, using gfun Maple package, we get linear recurrences:

Linear recurrences

$$
\begin{aligned}
&(125\, n^3 - 125\, n)\; e_n + \\
&(-475\, n^3 - 150\, n^2 + 325\, n)\; e_{n+1} + \\
&(-1625\, n^3 - 13650\, n^2 - 29125\, n - 17100)\; e_{n+2} + \\
&(5925\, n^3 + 65550\, n^2 + 204825\, n + 190800)\; e_{n+3} + \\
&(-10950\, n^3 - 149850\, n^2 - 609000\, n - 744300)\; e_{n+4} + \\
&(43599\, n^3 + 638460\, n^2 + 3028701\, n + 4633680)\; e_{n+5} + \\
&(-97781\, n^3 - 1680378\, n^2 - 9481237\, n - 17550960)\; e_{n+6} + \\
&(122749\, n^3 + 2388066\, n^2 + 15211685\, n + 31648968)\; e_{n+7} + \\
&(-184402\, n^3 - 3954630\, n^2 - 27717140\, n - 63149544)\; e_{n+8} + \\
&(280081\, n^3 + 6826380\, n^2 + 54868451\, n + 145130568)\; e_{n+9} + \\
&(-205649\, n^3 - 5654610\, n^2 - 51851989\, n - 158722620)\; e_{n+10} + \\
&(37439\, n^3 + 1339686\, n^2 + 16635271\, n + 70682784)\; e_{n+11} + \\
&(-68686\, n^3 - 3028038\, n^2 - 43616336\, n - 205972920)\; e_{n+12} + \\
&(222029\, n^3 + 9258780\, n^2 + 128417911\, n + 592399800)\; e_{n+13} + \\
&(-241115\, n^3 - 10519830\, n^2 - 152823475\, n - 739190880)\; e_{n+14} + \\
&(134151\, n^3 + 6201222\, n^2 + 95476551\, n + 489605640)\; e_{n+15} + \\
&(-42231\, n^3 - 2067834\, n^2 - 33729375\, n - 183277332)\; e_{n+16} + \\
&(7470\, n^3 + 386418\, n^2 + 6659316\, n + 38233296)\; e_{n+17} + \\
&(-678\, n^3 - 36972\, n^2 - 671670\, n - 4065240)\; e_{n+18} + \\
&(24\, n^3 + 1380\, n^2 + 26436\, n + 168720)\; e_{n+19} \;= 0.
\end{aligned}
$$

Initial values

$$
\begin{aligned}
e_0 &= 1,\\
e_1 &= 1,\\
e_2 &= 4,\\
e_3 &= 17,\\
e_4 &= 77,\\
e_5 &= 364,\\
e_6 &= 1776,\\
e_7 &= 8881,\\
e_8 &= 45296,\\
e_9 &= 234806,
\end{aligned}
$$

$$
\begin{aligned}
e_{10} &= 1233816,\\
e_{11} &= 6558106,\\
e_{12} &= 35202448,\\
e_{13} &= 190568779,\\
e_{14} &= 1039296373,\\
e_{15} &= 5704834700,\\
e_{16} &= 31494550253,\\
e_{17} &= 174759749005,\\
e_{18} &= 974155147162.
\end{aligned}
$$

Equation for closed closures

$$
\begin{aligned}
\mathcal{C}los_0 \quad ::= \quad & \mathcal{L}_0 \times \square \;\mid\; \mathcal{L}_1 \times \langle \mathcal{C}los_0 \rangle \;\mid \\
& \mathcal{L}_2 \times \langle \mathcal{C}los_0, \mathcal{C}los_0 \rangle \;\mid\; \mathcal{L}_3 \times \langle \mathcal{C}los_0, \mathcal{C}los_0, \mathcal{C}los_0 \rangle \;\mid\; \cdots \\
= \quad & \bigoplus_{m \geq 0} (\mathcal{L}_m \times \mathcal{C}los_0^m)
\end{aligned}
$$

Equation for closed closures

$$\begin{aligned}\mathcal{C}los_0 &::= \mathcal{L}_0 \times \square \mid \mathcal{L}_1 \times \langle \mathcal{C}los_0 \rangle \mid \\ & \quad \mathcal{L}_2 \times \langle \mathcal{C}los_0, \mathcal{C}los_0 \rangle \mid \mathcal{L}_3 \times \langle \mathcal{C}los_0, \mathcal{C}los_0, \mathcal{C}los_0 \rangle \mid \cdots \\ &= \bigoplus_{m \geq 0} (\mathcal{L}_m \times \mathcal{C}los_0^m)\end{aligned}$$

Then the equation for the generating function

$$C_0(z) = \sum_{m \geq 0} L_m(z) C_0(z)^m.$$

The smallest closed closures

size	closures	total
$0, 1$		0
2	$\langle \lambda \underline{0}, \square \rangle$	1
3	$\langle \lambda\lambda \underline{0}, \square \rangle$ $\langle \underline{0}, \langle \lambda \underline{0}, \square \rangle \rangle$	2
4	$\langle \lambda\lambda\lambda \underline{0}, \square \rangle$ $\langle \lambda\lambda \underline{1}, \square \rangle$ $\langle \lambda(\underline{0}\underline{0}), \square \rangle$ $\langle \lambda \underline{0}, \langle \lambda \underline{0}, \square \rangle \rangle$ $\langle \underline{0}, \langle \lambda\lambda \underline{0}, \square \rangle \rangle$ $\langle \underline{0}, \langle \underline{0}, \langle \lambda \underline{0}, \square \rangle \rangle \rangle$	6

▸ Plain closures

Theorem

Theorem

There exist

- $\overline{\rho} < \underline{\rho}$ *satisfying* $\overline{\rho} < \underline{\rho} < \rho_{L_\infty}$ *and*
- *functions* $\theta(n), \kappa(n)$ *satisfying* $\limsup_{n\to\infty} \sqrt[n]{\theta(n)} = \limsup_{n\to\infty} \sqrt[n]{\kappa(n)} = 1$

such that for sufficiently large n *we have*

- $\underline{\rho}^{-n}\theta(n) < c_{0,n} < \overline{\rho}^{\ -n}\kappa(n).$

Sketch of the proof (case $\underline{C}_0(z)$)

- We may think of bounding $C_0(z)$ below by

$$\underline{C}_0(z) = \sum_{m \geq 0} L_0(z)\underline{C}_0(z)^m = L_0(z) \sum_{m \geq 0} \underline{C}_0(z)^m = \frac{L_0(z)}{1 - \underline{C}_0(z)}.$$

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but this does not work.

- We approximate λ-terms by h-shallow λ-terms
  - terms with de Bruijn indices bounded by h.

The approximation works for $h = 153$

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  but this does not work.
- We approximate λ-terms by *h-shallow* λ-terms
  - terms with de Bruijn indices bounded by h.

  The approximation works for $h = 153$
  - not for $h = 152$!

▸ Generating functions

Sketch of the proof (case $\overline{C}_0(z)$)

- We bound $C_0(z)$ above by

$$\overline{C}_0(z) = \sum_{m\geq 0} L_\infty(z)\overline{C}_0(z)^m = L_\infty(z)\sum_{m\geq 0}\overline{C}_0(z)^m = \frac{L_\infty(z)}{1-\overline{C}_0(z)}.$$

Sketch of the proof (case $\overline{C}_0(z)$)

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- It works !

Conclusion

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- Counting plain environments and closed closures is relatively easy
  - This is not the most realistic model.
- Counting closed environments and closures is challenging.
  - This is what we want.
- The implicit and infinite specification of closed closures based on closed λ-terms complicates the quantitative analysis,

Any questions !

De Bruijn indices

Back

We want to count representatives of equivalence classes of λ-terms modulo α conversion.

Variables are replaced by natural numbers.

De Bruijn indices

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$\lambda x.(x\,x)$
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$\lambda x(x\,y)$
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$\lambda x.\lambda y.x$	$\lambda\lambda 1$
$\lambda x(x\, y)$	???
$(\lambda x.x)\,(\lambda x.\lambda y(y\, x))$	$(\lambda 0)\ (\lambda\lambda 0\ 1)$

Size of terms

▸ Back

- Abstractions λ are of size 1.
- Applications are of size 0.

Three methods (at least!) for counting de Bruijn indices:

- De Bruijn indices have size 0,
- De Bruijn indices have size 1
- De Bruijn indices n have size $n+1$

	Variable size 0	Variable size 1	natural size
$\lambda(0\ 0)$	2	4	4
$\lambda\lambda 1$	2	3	4
$(\lambda 0)\,(\lambda\lambda(0\ 1))$	5	8	9

The h-approximations of C_0

▸ Back

$$
\begin{aligned}
L_0^{(h)}(z) &= zL_1^{(h)}(z) + zL_0^{(h)}(z)L_0^{(h)}(z)\\
L_1^{(h)}(z) &= zL_2^{(h)}(z) + zL_1^{(h)}(z)L_1^{(h)}(z) + z\\
L_2^{(h)}(z) &= zL_3^{(h)}(z) + zL_2^{(h)}(z)L_2^{(h)}(z) + z + z^2\\
&\dots\\
L_{h-1}^{(h)}(z) &= zL_h^{(h)}(z) + zL_{h-1}^{(h)}(z)L_{h-1}^{(h)}(z) + z + z^2 + \dots + z^{h-1}\\
L_h^{(h)}(z) &= zL_h^{(h)}(z) + zL_h^{(h)}(z)L_h^{(h)}(z) + z + z^2 + \dots + z^h
\end{aligned}
$$