On the number of variables in special classes of random lambda-terms

B. Gittenberger[1] I. Larcher[1]

[1]Institute of Discrete Mathematics and Geometry
Technische Universität Wien

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Lambda-terms

The set Λ of lambda-terms is defined by the following grammar:

- x variable $\Rightarrow x \in \Lambda$.
- $M, N \in \Lambda \Rightarrow MN \in \Lambda$. $\rightarrow$ *application*
- x variable and $M \in \Lambda \Rightarrow \lambda x.M \in \Lambda$. $\rightarrow$ *abstraction*

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$\rightarrow$ Variables that are not bound by any λ are called free.

Lambda-DAGs

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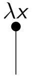

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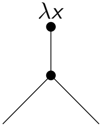

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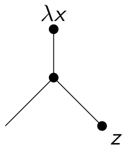

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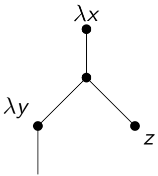

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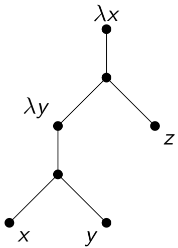

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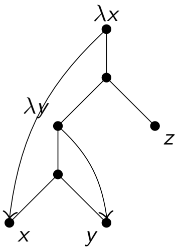

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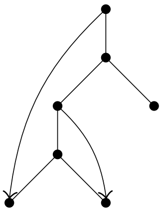

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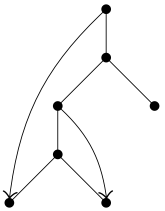

- α-equivalence

Lambda-DAGs

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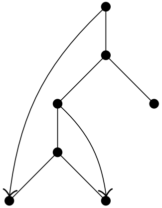

- α-equivalence
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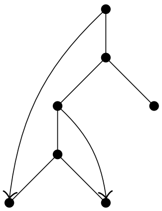

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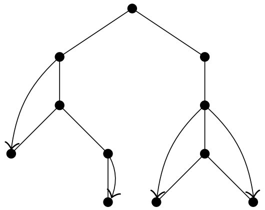

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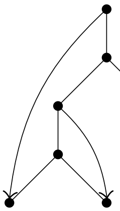

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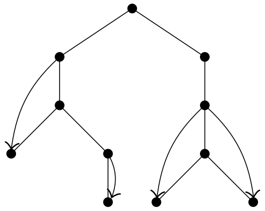

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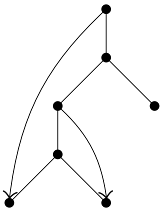
$|x| = 1 \qquad |MN| = 1 + |M| + |N| \qquad |\lambda x.M| = 1 + |M|$

Lambda-DAGs

$|\lambda x.((\lambda y.(xy))z)| = 7$



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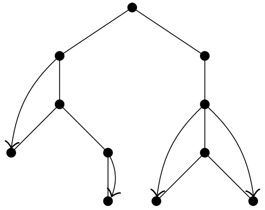

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- **Generating functions:**

$a_n \ldots$ number of lambda-terms of size n

$A(z) = \sum_{n \geq 0} a_n z^n \ldots$ GF of $(a_n)_{n \geq 0}$

$\Rightarrow [z^n]A(z) = a_n$

- **Generating functions:**

$a_{n,k}$ ... number of lambda-terms of size n with k variables

$A(z,u) = \sum_{n,k\geq 0} a_{n,k} z^n u^k$ ... bivariate GF of $(a_{n,k})_{n\geq 0, k\geq 0}$

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- **Singularity analysis:**

  ρ ... dominant singularity of $A(z)$ of type $-\alpha$ (*i.e.*, $(1-\frac{z}{\rho})^{-\alpha}$)

$$[z^n]A(z) \sim \rho^{-n} \frac{n^{\alpha-1}}{\Gamma(\alpha)} \qquad \text{as } n \longrightarrow \infty$$

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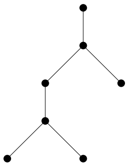
- **Symbolic method:**

$$\mathcal{C} = \mathcal{A} \cup \mathcal{B} \qquad \Rightarrow C(z) = A(z) + B(z)$$
$$\mathcal{C} = \mathcal{A} \times \mathcal{B} \qquad \Rightarrow C(z) = A(z) \cdot B(z)$$

Reason for super exponential growth: possible variable bindings

$\rightarrow$ Comparison to Motzkin trees: $M_n = \sqrt{\frac{3}{4\pi n^3}}3^n + O(3^n n^{-2})$



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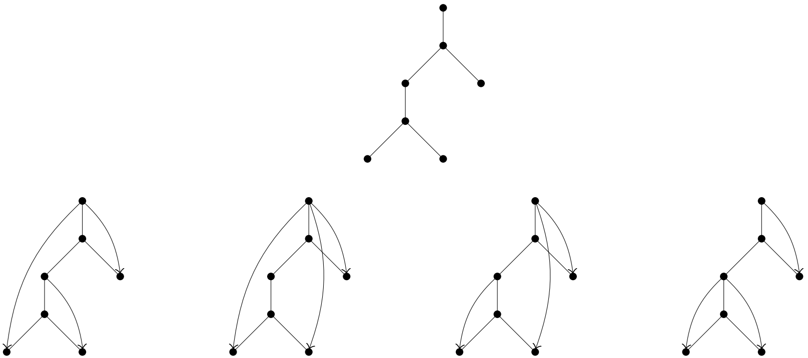

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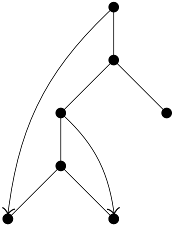

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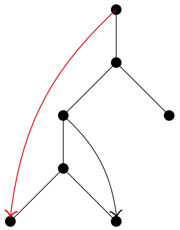

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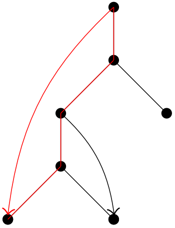

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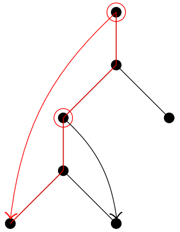

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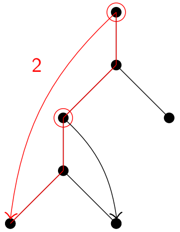

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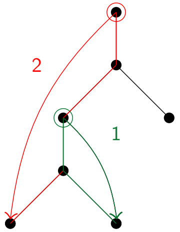

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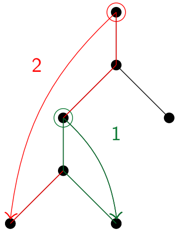

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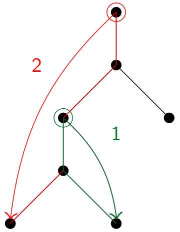

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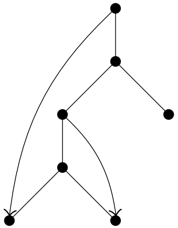

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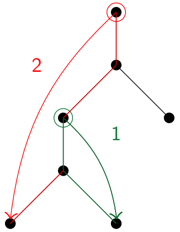

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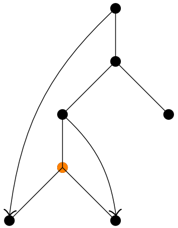

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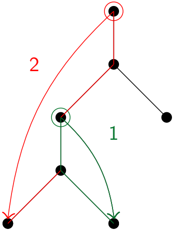

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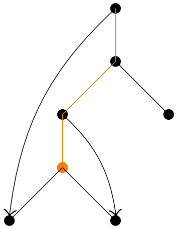

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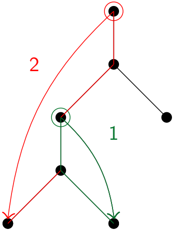

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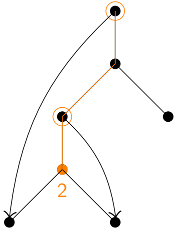

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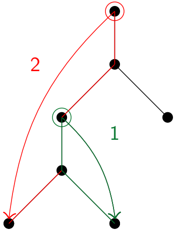

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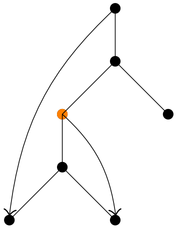

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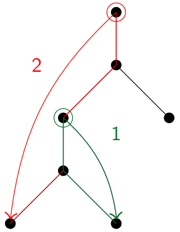

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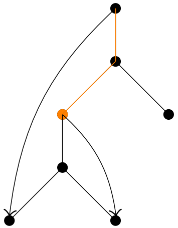

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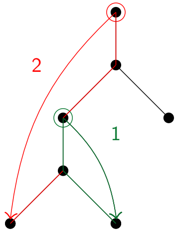

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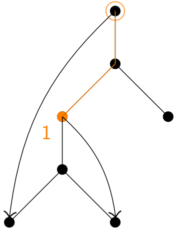

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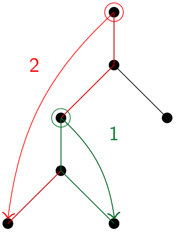

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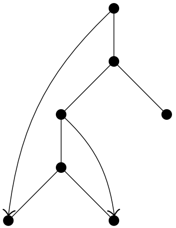
Unary height $h_u(t)$ of a lambda-term =
maximum number of unary nodes occurring in the branches of the underlying Motzkin tree



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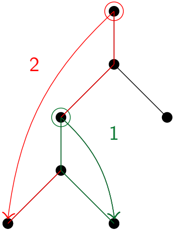

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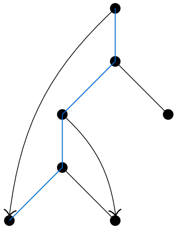

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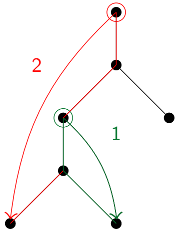

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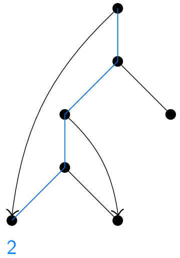

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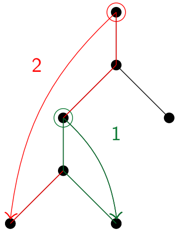

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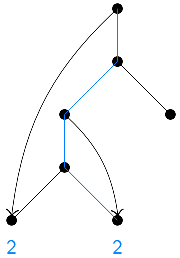
maximum number of unary nodes occurring in the branches of the underlying Motzkin tree



Unary length of a binding of a variable e by some abstraction λe =
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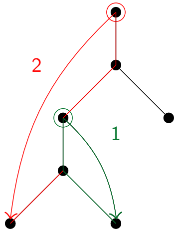

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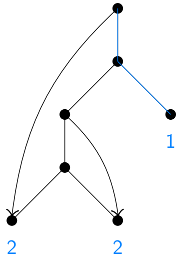

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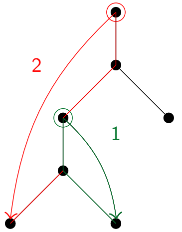

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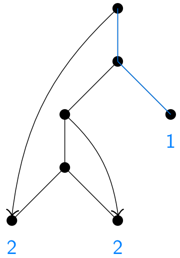
$$h_u(t) = 2$$



Lambda-terms with bounded unary length of each binding

$\mathcal{G}_k \ldots$ class of closed lambda-terms where all bindings have unary length at most k

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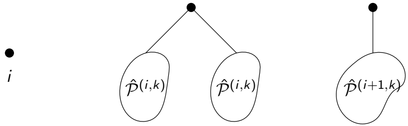

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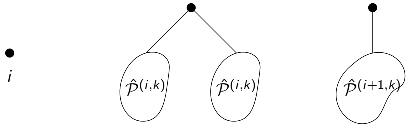

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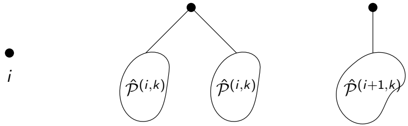

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$$\hat{P}^{(k,k)}(z) = kz + z\hat{P}^{(k,k)^2}(z) + z\hat{P}^{(k,k)}(z)$$

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Theorem (Bodini, Gardy, Gittenberger, Gołębiewski, 2018)

- *The dominant singularity* $\hat{\rho}_k$ *of* $G_k(z)$ *comes from* $\hat{R}_1(z) = (1-z)^2 - 4kz^2$, *i.e.,* $\hat{\rho}_k = \frac{1}{1+2\sqrt{k}}$, *and is of type* $\frac{1}{2}$
- *For* $n \longrightarrow \infty$

$$[z^n]G_k(z) = g_k(1+2\sqrt{k})^n n^{-\frac{3}{2}} \left(1 + O\left(\frac{1}{n}\right)\right),$$

with $g_k = \sqrt{\frac{\sqrt{k}+2k}{4\pi \prod_{l=2}^{k+1} c_l}}$,

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$$\boxed{G_k(z,u) = \hat{P}^{(0,k)}(z,u) = \frac{1-\sqrt{\hat{R}_{k+1}(z,u)}}{2z}}$$

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Theorem (Quasi-Power Theorem, Hwang, 1998)

Let X_n be a sequence of random variables with the property that

$$\frac{[z^n]G(z,u)}{[z^n]G(z,1)} \sim A(u)B(u)^{\lambda_n},$$

with $\lambda_n \to \infty$.
Set $\mu = B'(1)$ and $\sigma^2 = B''(1) + B'(1) - B'(1)^2$. If $\sigma^2 \neq 0$, then

$$\frac{X_n - \mathbb{E}X_n}{\sqrt{\mathbb{V}X_n}} \to \mathcal{N}(0,1), \qquad \text{as } n \to \infty,$$

with

$$\mathbb{E}X_n \sim \mu\lambda_n \qquad \text{and} \qquad \mathbb{V}X_n \sim \sigma^2\lambda_n.$$

$$\frac{[z^n]G_k(z,u)}{[z^n]G_k(z,1)} = \left(\frac{1+2\sqrt{ku}}{1+2\sqrt{k}}\right)^n \sqrt{\frac{\sqrt{ku}+2ku}{2k+\sqrt{k}} \prod_{j=2}^{k+1} \frac{c_j(1)}{c_j(u)}} \left(1+O\left(\frac{1}{n}\right)\right)$$

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Theorem (Gittenberger, L., 2018)

Let X_n be the number of variables in closed lambda-terms where all bindings have unary length at most k. Then X_n is asymptotically normally distributed with asymptotic mean and variance given by

$$\mathbb{E}X_n \sim \frac{k}{\sqrt{k}+2k} n \quad \text{and} \quad \mathbb{V}X_n \sim \frac{k^2}{2\sqrt{k}(\sqrt{k}+2k)^2} n.$$

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For $k \longrightarrow \infty$: $\quad \mathbb{E}X_n \longrightarrow \frac{n}{2} \quad$ and $\quad \mathbb{V}X_n \longrightarrow 0.$

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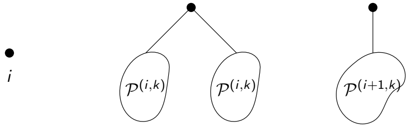

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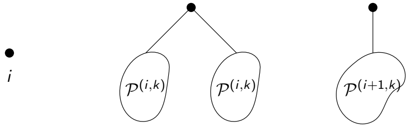

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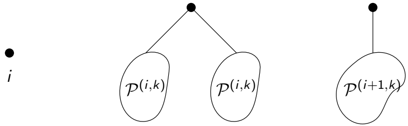

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Auxiliary sequences:

$$u_0 = 0, \quad u_{i+1} = u_i^2 + i + 1 \quad \text{for } i \geq 0,$$

and

$$N_i = u_i^2 - u_i + i, \quad \text{for all } i \geq 0.$$

j	N_j	u_j
1	1	1
2	8	3
3	135	12
4	21760	148
5	479982377	21909
⋮	⋮	⋮

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Theorem (Bodini, Gardy, Gittenberger, Gołębiewski, 2018)

Let ρ_k be the dominant singularity of $H_k(z)$.

(i) *If $N_j < k < N_{j+1}$, then ρ_k comes from R_{j+1} and*

$$[z^n]H_k(z) \sim h_k n^{-3/2} \rho_k^{-n}, \text{ as } n \to \infty.$$

(ii) *If $k = N_j$, then $\rho_k = \frac{1}{2u_j}$ and*

$$[z^n]H_k(z) \sim h_{N_j} n^{-5/4} \rho_k^{-n} = h_{N_j} n^{-5/4} (2u_j)^n, \text{ as } n \to \infty.$$

$$\boxed{H_k(z) = P^{(0,k)}(z) = \frac{1-\sqrt{R_{k+1}(z)}}{2z}}$$

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$$R_{1,k}(z) = 1 - 4kz^2,$$

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Total number of variables

Theorem (Gittenberger, L., 2018)

Let $B(u) = \frac{\rho_k(u)}{\rho_k(1)}$. *If* $B''(1) + B'(1) - B'(1)^2 \neq 0$, *then the total number* X_n *of variables in closed lambda-terms with unary height at most* k *is asymptotically normally distributed with*

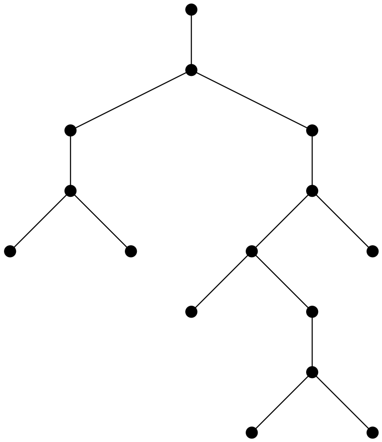
$$\mathbb{E}X_n \sim \mu \cdot n \qquad \text{and} \qquad \mathbb{V}X_n \sim \sigma^2 \cdot n,$$

where $\mu = B'(1)$ *and* $\sigma^2 = B''(1) + B'(1) - B'(1)^2$.

bound k	$j+1$	$B''(1)+B'(1)-B'(1)^2$	$B'(1)$
1	**2**	**0**	**0**
2	2	0.0385234386	0.4381229337
3	2	0.0210625856	0.4414407371
4	2	0.0167136805	0.4463973717
5	2	0.0148700270	0.4504258849
6	2	0.0138224393	0.4536185043
7	2	0.0131157948	0.4561987871
8	**3**	**0.048**	**0.4**
9	3	0.0582322465	0.4566104777
10	3	0.0470481360	0.4560418340
11	3	0.0396601986	0.4560810348
12	3	0.0345090124	0.4564489368
⋮	⋮	⋮	⋮
133	3	0.0077469541	0.4821900098
134	3	0.0077234960	0.4822482745
135	**4**	**0.0108490182**	**0.4782608696**

Number of variables in the separate unary levels

The *i*-**th unary level** contains all vertices with unary height *i*.



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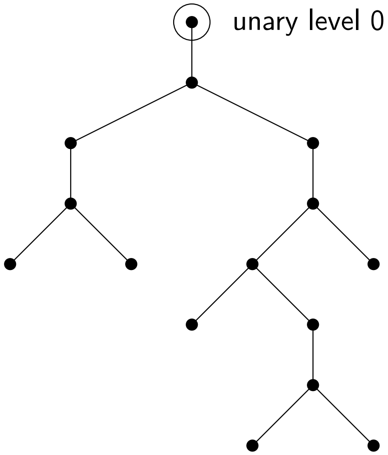

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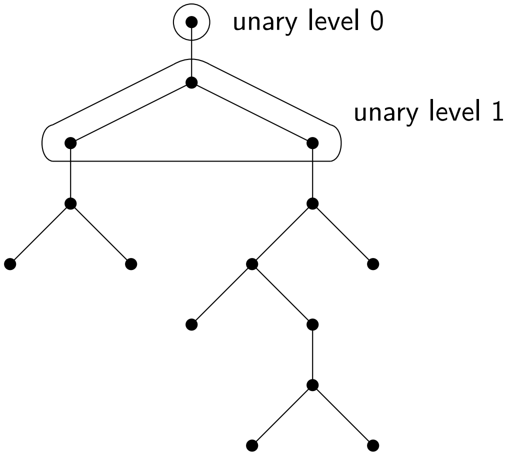

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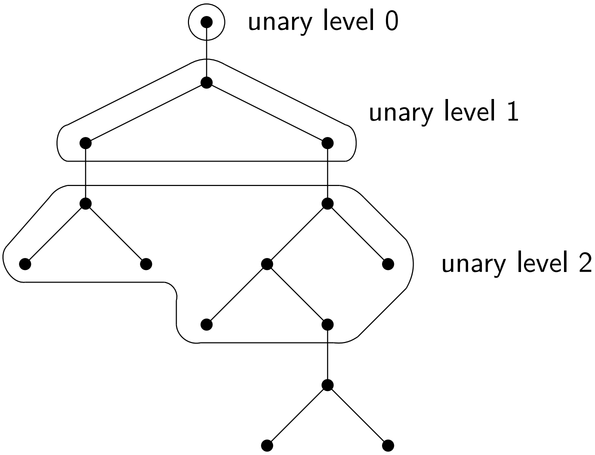

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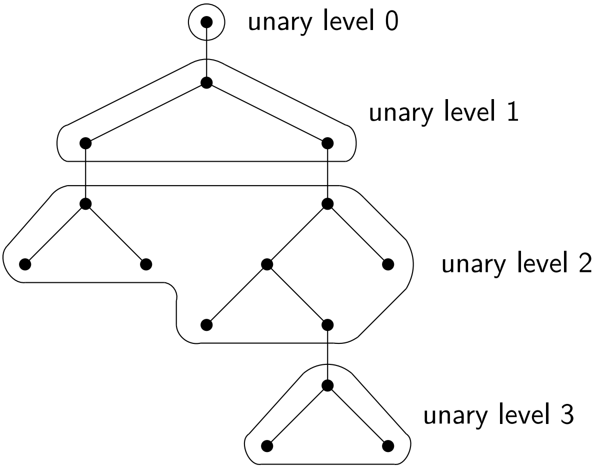

$C(z,u)\ldots$ GF of the class of binary trees where z marks the size and u the number of leaves $\to$ $C(z,u) = \frac{1-\sqrt{1-4uz^2}}{2z}$

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Generating functions of lambda-terms with unary height at most k:

$H_k(z)\ldots$ z marks the size

$$H_k(z) = C(z, C(z, 1 + \ldots + C(z, (k-l) + \ldots + C(z, (k-1) + C(z,k))) \ldots) \ldots))$$

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$H_k(z)\ldots$ z marks the size

$H_k(z,u)\ldots$ z marks the size, u marks the total number of leaves

${}_{k-l}H_k(z,u)\ldots$ z marks the size, u marks the number of leaves on the $(k-l)$-th unary level $(0 \leq l \leq k)$

$$H_k(z) = C(z, C(z, 1 + \ldots + C(z, (k-l) + \ldots + C(z, (k-1) + C(z,k)))\ldots)\ldots))$$

$$H_k(z,u) = C(z, C(z, u + \ldots + C(z, (k-l)\cdot u + \ldots + C(z, (k-1)\cdot u + C(z, k\cdot u)))\ldots)\ldots))$$

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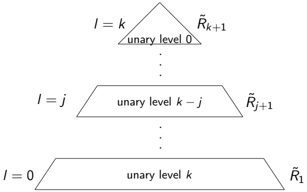
$$\boxed{{}_{k-l}H_k(z,u) = \frac{1-\sqrt{\tilde{R}_{k+1}(z,u)}}{2z}}$$



where

$$\tilde{R}_1(z,u) = 1 - 4z^2k,$$

and for $2 \le i \le k+1, i \neq l+1$

$$\tilde{R}_i(z,u) = 1 - 4z^2(k-i+1) - 2z + 2z\sqrt{\tilde{R}_{i-1}(z,u)},$$

and

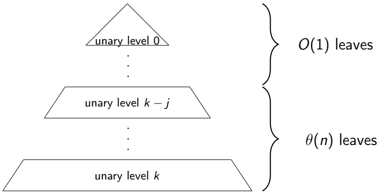
$$\tilde{R}_{l+1}(z,u) = 1 - 4z^2u(k-l) - 2z + 2z\sqrt{\tilde{R}_{l-1}(z,u)}.$$

Number of leaves on the separate unary levels

Let $k \in (N_j, N_{j+1})$

- $l > j : \mathbb{E}X_n \sim C_{k,l}$
- $l \leq j : \mathbb{E}X_n \sim \tilde{C}_{k,l} \cdot n$

with constants $C_{k,l}$ and $\tilde{C}_{k,l}$.



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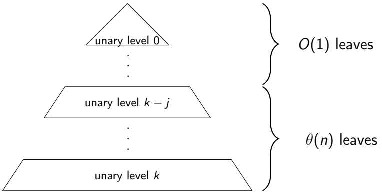
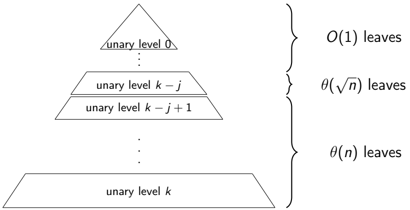


Let $k = N_j$

- $l > j : \mathbb{E}X_n \sim D_{k,l}$
- $l = j : \mathbb{E}X_n \sim \hat{D}_{k,l} \cdot \sqrt{n}$
- $l < j : \mathbb{E}X_n \sim \tilde{D}_{k,l} \cdot n$

with constants $D_{k,l}$, $\hat{D}_{k,l}$ and $\tilde{D}_{k,l}$.



Number of leaves on the separate unary levels

Let $k \in (N_j, N_{j+1})$

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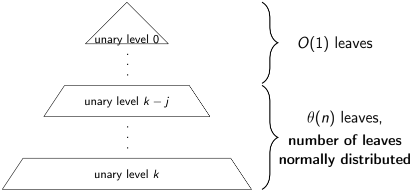

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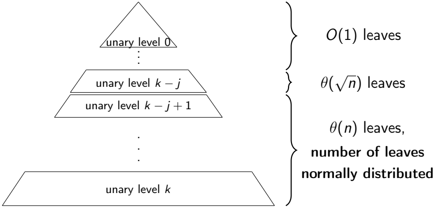
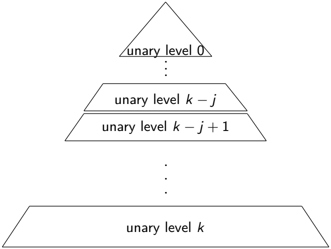
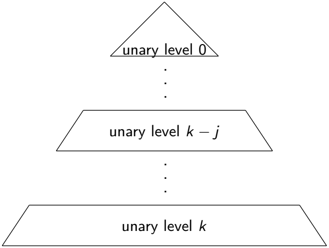


“Unary profile”

$$k = N_j$$



$$N_j < k < N_{j+1}$$



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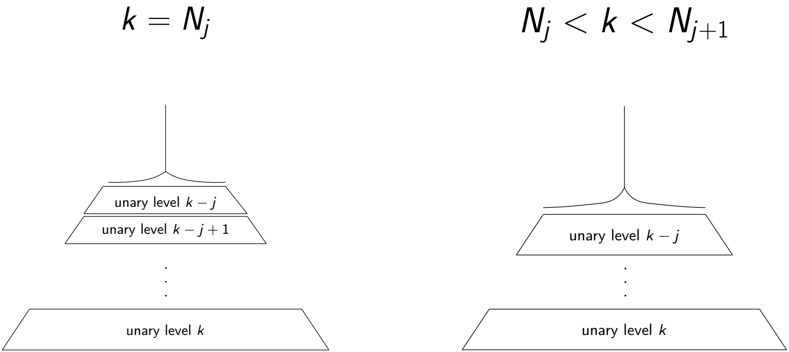

Thank you!