Statistical properties of lambda terms

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CLA, Jussieu, Paris, 24/05/2018

Chapter 1. *Historical overview*

What is a closed lambda term?

Expression such as

$$\lambda x.\lambda y.(\lambda z.(\lambda x.zx)(\lambda y.zy))(xy)$$

Expression such as

$$\lambda x.\lambda y.(\lambda z.(\lambda x.zx)(\lambda y.zy))(xy)$$

- $\lambda -$ abstraction

Expression such as

$$\lambda x.\lambda y.(\lambda z.(\lambda x.z @ x)(\lambda y.z @ y)) @ (x @ y)$$

- λ – abstraction
- @ – application

Expression such as

$$\lambda x.\lambda y.(\lambda z.(\lambda x.zx)(\lambda y.zy))(xy)$$

- λ — abstraction
- @ — application
- x, y, z — variables

also lambda terms are..

Unary-binary trees with links

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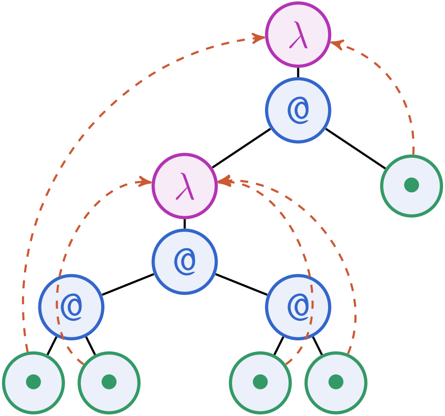

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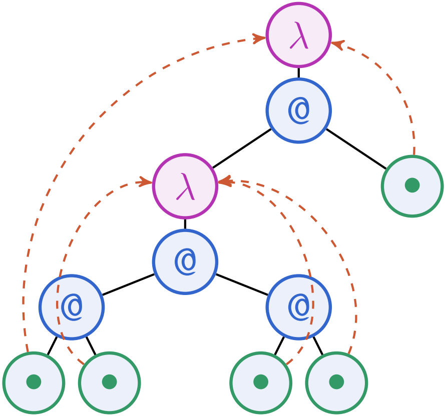

- λ are *abstractions*
- @ are *applications*
- • are *variables*

Local summary

Closed lambda terms are:

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- Expressions such as $\lambda x.\lambda y.(\lambda z.(\lambda x.zx)(\lambda y.zy))(xy)$
- Unary-binary trees with links
- TODO: What are plain lambda terms?
- TODO: What is the difference between closed and plain lambda terms?

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Motivation

First application. Software testing techniques.

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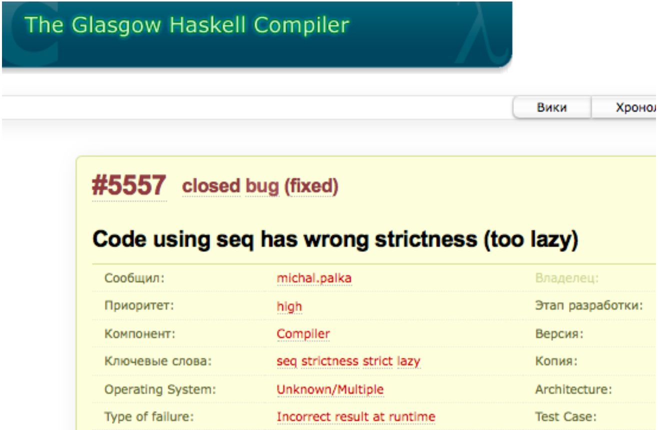

Second motivation. Relation between (linear) lambda terms and (trivalent) maps.

Third motivation. Development of analytic combinatorics.

Motivation

Summary

- Software testing techniques
- Lambda terms vs. maps
- Development of analytic combinatorics

Plain and closed lambda terms

- **Closed terms**: unary-binary trees with links between variables and abstractions
- **Plain terms**: unary-binary trees with some variables unlinked

Example.

Example.

- $\lambda z.(\lambda y.zy)$ is a **closed** term

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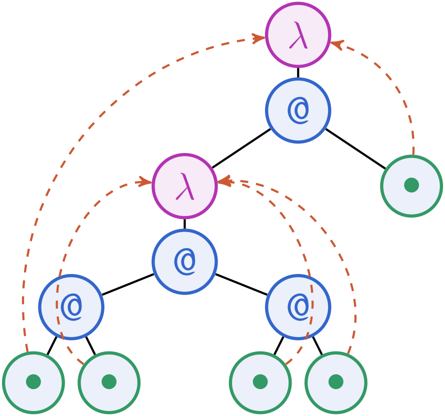
- $\lambda z.(\lambda y.zy)$ is a **closed** term
- $\lambda y.zy$ is a **plain** term

Example.

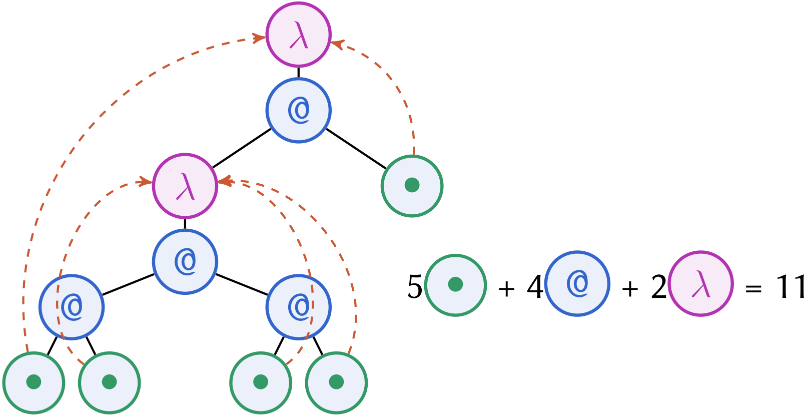
- $\lambda z.(\lambda y.zy)$ is a **closed** term
- $\lambda y.zy$ is a **plain** term
- $(\lambda x.x)(\lambda y.xy)$?

Size notion of a lambda term

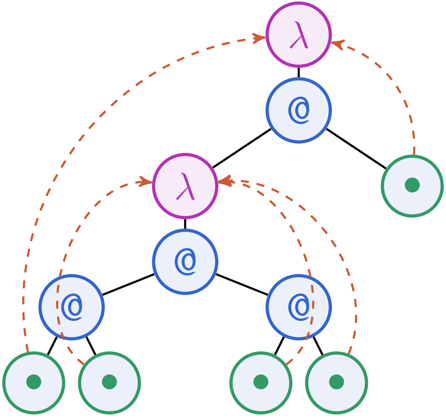
What is the size of



First approach. Total number of nodes.

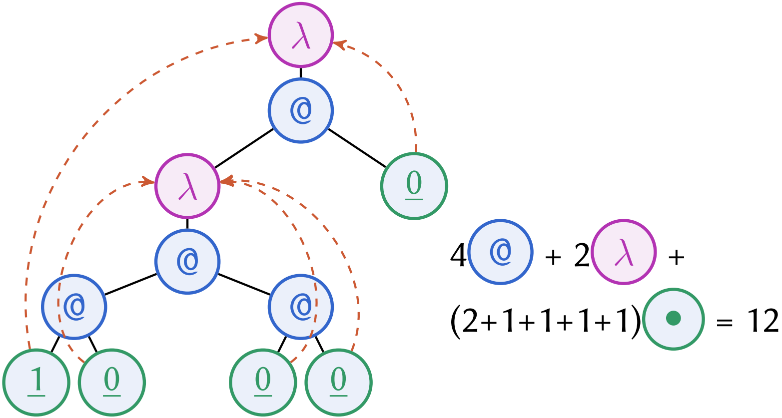


Second approach. Only abstractions and applications.

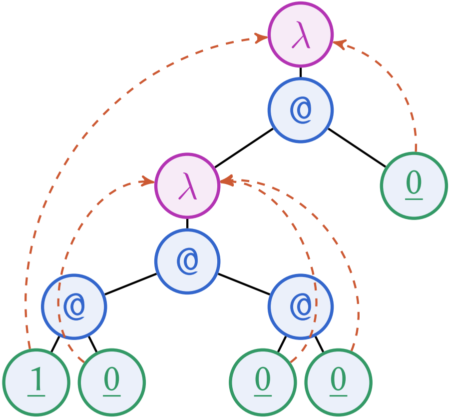


4 @ + 2 λ = 6

Third approach. Natural counting.



Fourth approach. Generalised natural counting.



4a @ +2b λ +5c • +

(1+0+0+0+0)d •

= 4a + 2b + 5c + d

Size notions, recap

- Variable size = 1
- Variable size = 0
- Variable size $\geqslant 1$ (Natural counting)
- Generalised natural counting

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In-detail analysis of size notions

1. Variable size equal 1

Variable size equal 1

- [Bodini, Gardy, Gittenberger, Jacquot '11+]
- Linear closed lambda terms
- Upper and lower bounds for asymptotics of all closed terms
- New bijections
- Continued [Zeilberger, Giorgetti + '15+]
- :(Formal power series is divergent

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2. Variable size equal 0

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- [Grygiel, Lescanne ’12]
- Enumeration and random generation
- [David, Grygiel, Kozik, Raffalli, Theyssier, Zaionc ’13]
- $\Theta(\sqrt{n/\log n})$ head abstractions
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3. Natural counting

Natural counting

- Variable size = unary distance to binding λ
- [Bendkowski, Grygiel, Lescanne, Zaionc '12]
- [Bodini, Gittenberger, Gołębiewski '16]
  Enumeration and asymptotics
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Why consider a simpler model?

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Main reason: advantages in random generation

Why consider a simpler model?

Marking variables imply control over expectations

Why consider a simpler model?

Another reason: parameter study

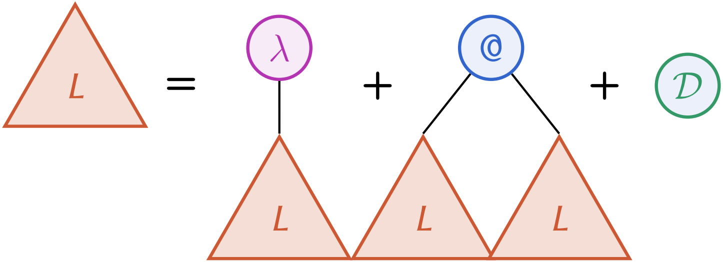
Why consider a simpler model?

Marking variables allow to study distributions

Chapter 2. *Statistical properties*

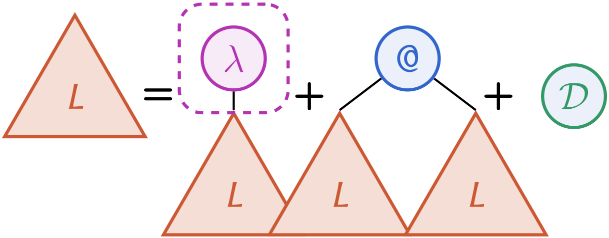
Plain lambda terms

Generating function for plain lambda terms



$$L(z) = zL(z) + zL^2(z) + \frac{z}{1-z}$$

Abstractions in plain terms?



$$L(z,u) = z{\color{purple}u}L(z,u) + zL(z,u) + \frac{z}{1-z}$$

$$L(z,u) \sim a(z,u) - b(z,u)\sqrt{1 - \frac{z}{\rho(u)}}$$

Multivariate Central Limit Theorem

Step 1. Extract coefficient of $L(z, u)$

$$L(z, u) \sim a(z, u) - b(z, u)\sqrt{1 - \frac{z}{\rho(u)}}$$

Step 2. Asymptotic behaviour of probability generating function

$$p_n(u) \sim A(u)B(u)^n$$

Step 3. Gaussian approximation from $A(u)B(u)^n$.

Applying multivariate CLT

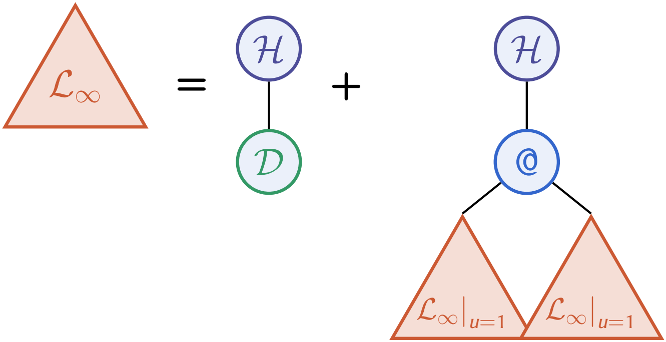
Theorem Joint Gaussian distribution for

- number of abstractions
- number of variables
- number of *redexes*

in plain lambda terms

Discrete distributions in plain terms

Head abstractions



$$L(z, u) = \frac{1}{1 - zu} \left(\frac{z}{1 - z} + zL(z, 1)^2 \right)$$

Theorem The following statistics follow discrete (geometric) limiting distributions

- The number of head abstractions
- Randomly chosed value of de Bruijn index

Chapter 3. *Infinite systems*

Drmota–Lalley–Woods theorem

Drmota–Lalley–Woods theorem

- Let $F(z)$ be a generating function
- Suppose it satisfies

$$F(z) = \Phi(F(z), z)$$

with Φ having combinatorial origin
- Then,

$$F(z) \sim a - b\sqrt{1 - \frac{z}{\rho}}$$

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What happens with infinite systems?

- [Drmota, Gittenberger, Morgenbesser '12]
- *If* Jacobian is a sum of identity matrix and a compact operator
- *And* specification is strongly connected
- *Then* Infinite-dimensional version holds

Closed lambda terms satisfy an infinite system

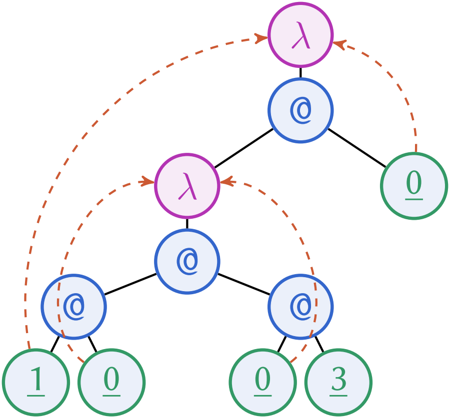
$$
\begin{aligned}
L_0(z) &= zL_1(z) + zL_0(z)^2 \quad , \\
L_1(z) &= zL_2(z) + zL_1(z)^2 + z \quad , \\
L_2(z) &= zL_3(z) + zL_2(z)^2 + z + z^2 \quad , \\
&\dots \\
L_\infty(z) &= zL_\infty(z) + zL_\infty(z) + \frac{z}{1-z} \quad .
\end{aligned}
$$

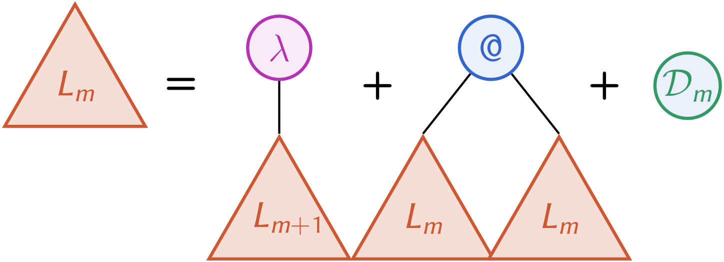
Why?

Adding m abstractions on the top of L_m makes the term closed

Example.

Example.





$$L_m(z) = zL_{m+1}(z) + zL_m^2(z) + z\frac{1 - z^m}{1 - z}$$

However

This system

$$
\begin{aligned}
L_0(z) &= zL_1(z) + zL_0(z)^2 \quad , \\
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&\cdots \\
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\end{aligned}
$$

Doesn’t satisfy infinite DLW theorem

New “master theorem”

Master theorem

Motivation

Master theorem

Motivation

$$L_m(z) = zL_{m+1}(z) + zL_m(z)^2 + \underbrace{z\frac{1-z^m}{1-z}}_{D_m} ,$$

$$L_\infty(z) = zL_\infty(z) + zL_\infty(z)^2 + \underbrace{z\frac{1}{1-z}}_{D_\infty} .$$

The difference $D_\infty - D_m$ is exponentially small

Master theorem

More generally.

$$L_m(z) = K_m(L_m, L_{m+1}, z, u) \;\;,$$
$$L_\infty(z) = K_\infty(L_\infty, L_\infty, z, u) \;\;,$$

$K_\infty - K_m$ is exponentially small specified at L_∞
L_m, K_m, u are vectors

Master theorem

Assumptions

1. The limiting system has Puiseux expansion
2. Limiting system dominates *m*-th system
3. The difference is exponentially small

Statement

- Each L_m also has Puiseux expansion*

Each L_m also has Puiseux expansion*

Two possible variants:

1. (Weak)
$$[z^n]L_m(z,u) \sim [z^n]\left(a(u) - b(u)\sqrt{1 - \frac{z}{\rho(u)}}\right)$$
2. (Strong)
The analytic continuation of $L_m(z,u)$ in a *delta-domain* satisfies

$$L_m(z,u) \sim a(u) - b(u)\sqrt{1 - \frac{z}{\rho(u)}}$$

Lambda terms Infinite systems

Applications of “master theorem”

Theorem Joint Gaussian distribution for

- number of abstractions
- number of variables
- number of successors
- number of *redexes*

in closed lambda terms

Theorem Discrete distributions for

- number of head abstractions
- randomly chosed index value
- redex search time
- free variables
- missing top abstractions

in <u>closed</u> lambda terms

Theorem Rayleigh distributions for

- unary height profile
- natural height profile

in plain and closed lambda terms for variables, abstractions and applications

Summary for statistics

- Normal limit laws
- New discrete limit laws
- Rayleigh limit laws

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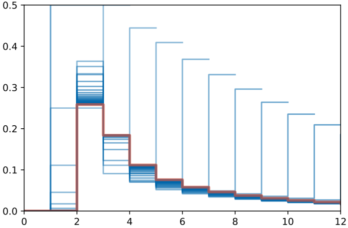
Grand summary for statistics

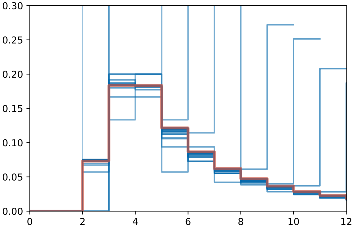
Parameter	Mean, $\sim$ plain	Mean, $\sim$ closed	Distribution plain	Distribution closed
Variables	$0.307n$		Normal	
Abstractions	$0.258n$		Normal	
Successors	$0.129n$		Normal	
Redexes	$0.091n$		Normal	
Index value	0.420		Geometric	
Redex search time	6.222	6.054	Discrete	Discrete
Head abstractions	0.420	1.447	Geometric	Discrete
m-openness	2.019	0	Discrete	trivial
Free variables	5.722	0	Discrete	trivial
Unary height profile	$0.122\sqrt{n}$		Rayleigh	
Natural height profile	$0.412\sqrt{n}$		Rayleigh	

Epilogue. *Open problems and experiments*

Redex search time distribution

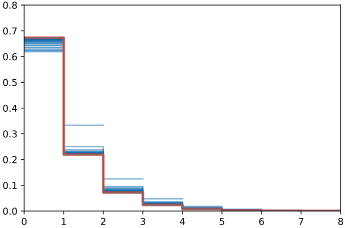


plain terms



closed terms

Head abstractions



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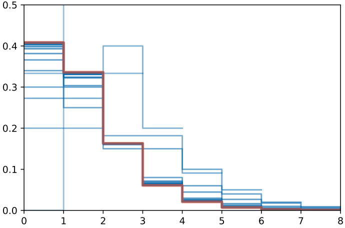

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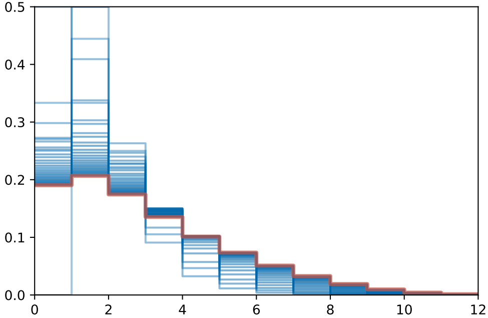
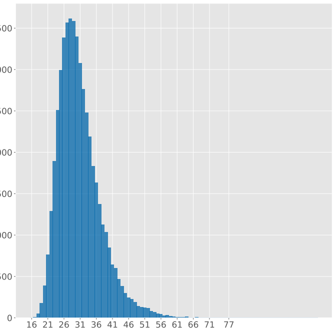
Free variables



Maximum number of variables bound to a single abstraction



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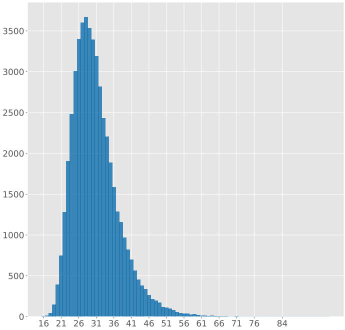
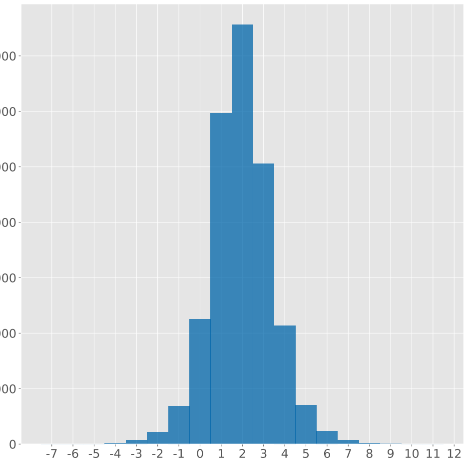
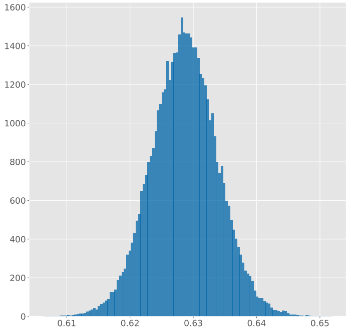


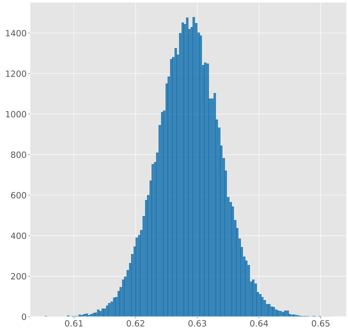
closed terms

Generalised m-openness



Number of binding abstractions





plain terms

closed terms

Open questions

- Maximum number of variables bound to a single abstraction
- Number of binding abstractions
- Generalised m-openness
- Number of BCI, BCK terms in natural size notion
- Average shape of a random lambda term after k β-reductions?

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Conclusions

First.

First. Many tractable parameters in closed lambda terms in de Bruijn size notion

Second.

Second. Efficient multiparametric Boltzmann sampling, tweaking the desired parameters

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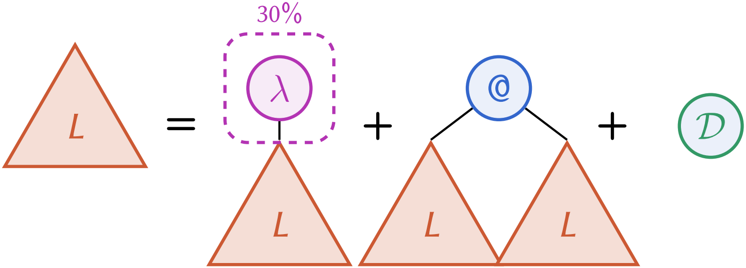

Third.

Third. Infinite systems of algebraic equations*

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Conjecture Premises of DLW Theorem for infinite strongly connected systems are not sufficient.

Thank you !