Combinatorics of explicit substitutions

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Outline

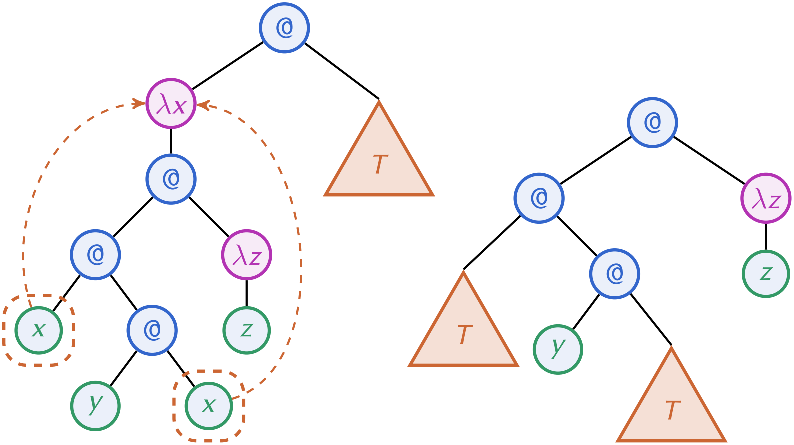
λ-calculus and substitution resolution

$\lambda\upsilon$-calculus and explicit substitutions

Our contribution

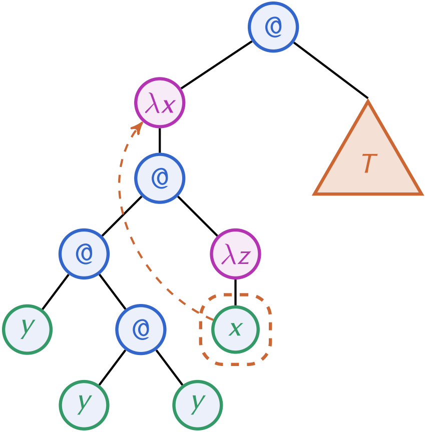
β-reduction in λ-calculus

Substitution of terms for variables forms the essence of β-reduction in λ-calculus. For instance, $(\lambda x.x(yx)(\lambda z.z))T \to_\beta T(yT)(\lambda z.z)$:



β-reduction in λ-calculus (ii)

However, what if we substitute T with *free variables* under an abstraction? For instance, what if z occurs freely in T, as in the following example:



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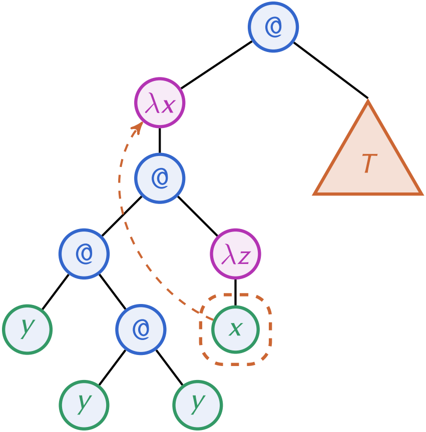

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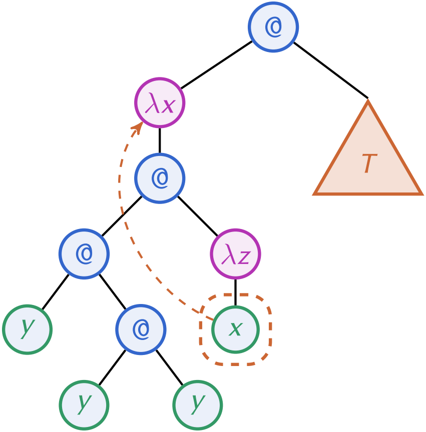

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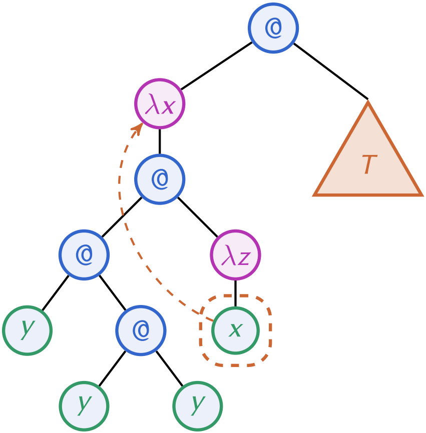

- We want to avoid the *capture* of variable z in T;
- Requires keeping track of variable names and (sometimes) their *renaming*;
- In principle feasible, but *in practice* it is not a trivial operation to execute.

β-reduction and computational effectiveness

Postulate (essentially due to de Bruijn '78)

The number of β-reductions required to normalise a given term (in other words, evaluate the encoded computation) *does not* quite reflect the computational effort required to *carry out* the computation.

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- and finally, it hides the details of *evaluation strategies*.

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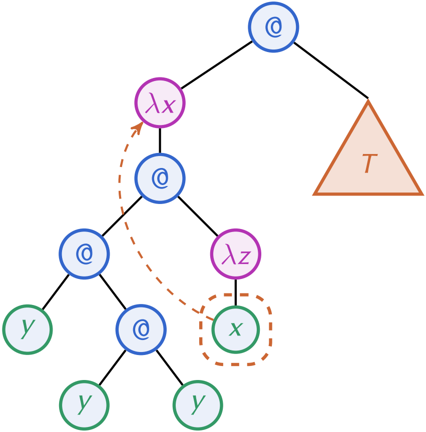
. . . and that is due to the <u>epitheoretic substitution</u> operation.

β-reduction and computational effectiveness (ii)

Issue I

β-reduction hides the non-trivial *details* of capture-avoiding substitution. Variable renaming can take time linear in the term size.

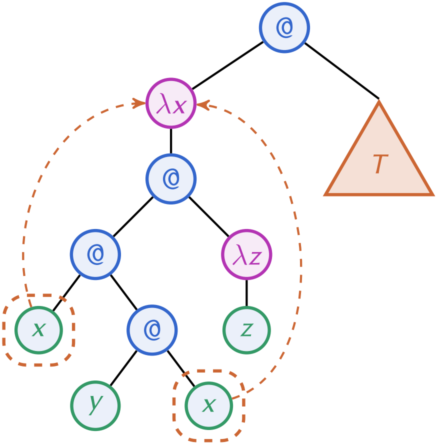


β-reduction and computational effectiveness (iii)

Issue II

β-reduction does not reflect the cost of *substitution resolution*.

Variable search can take time linear in the term size.



β-reduction and computational effectiveness (iv)

Issue III

β-reduction hides the details of *evaluation strategies*. What if the substitution is carried out non-strictly, e.g. it is suspended and evaluated *on demand* (perhaps even never)?

Real-life[1] example of an infinite lists of Fibonacci numbers:

```
fibs = 0 : 1 : next fibs
  where
    next (a : b : xs) = (a + b) : next (b : xs)
```

[1] Is this the *real life*? Is this just fantasy? (. . .) [Mercury et al. '75]

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Quest

Investigate quantitative aspects of substitution. For instance,

- What is the average-case cost of resolving substitutions in computations (terms) of size n? How does it change depending on the assumed evaluation strategy? What contributes to its execution time?
- What is the *average-case complexity* of abstract machines executing (terminating) computations of considered calculi? Is it possible to optimise them based on the structure of typical computations?

Reflections on combinatory logic

Normal-order reduction

If a term (combinator) T is normalisable, then the iterative contraction of the leftmost-outermost redex leads to the (unique) normal form of T.

$$
\begin{array}{lrcl}
Sxyz \to xz(yz) & R_0 & = & S \oplus K \oplus KR_0 \oplus SR_0 \oplus SR_0R_0 \\
Kxy \to x & R_1 & = & SR_1 \oplus KR_1 \oplus SR_0R_1 \oplus SR_1R_0 \oplus KR_0C \oplus \\
 & & & KSCR_0 \oplus KKCR_0 \oplus KSCR_0R_0 \oplus K(SR_0)CR_0 \\
 & & & \oplus SSSR_0 \oplus SSKR_0 \oplus SS(SR_0)R_0 \\
 & R_2 & = & \ldots
\end{array}
$$

Figure: Rewriting rules of SK-combinators and the corresponding normal-order reduction grammars.

Reflections on combinatory logic (ii)

Theorem (B., Grygiel and Zaionc '17)

For each $k \geq 1$, the asymptotic density $\mu(R_k/C)$ of combinators reducing in k normal-order reduction steps in the set of all combinators is *positive*.

In particular, we have:

k	$\mu(R_k/C)$
0	0.
1	0.08961
2	0.06417
3	0.05010
4	0.04131
5	0.03570
6	0.03119
7	0.02798

Problem

How to *port* these results to the realm of λ-calculus having an *external* substitution?

... use *explicit substitutions*!

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$\lambda\upsilon$-calculus (lambda upsilon calculus)

A simplistic calculus of explicit substitutions due to Lescanne '94.

$$\mathcal{T} ::= \mathcal{N} \mid \lambda\mathcal{T} \mid \mathcal{T}\mathcal{T} \mid \mathcal{T}[\mathcal{S}]$$
$$\mathcal{S} ::= \mathcal{T}/ \mid \Uparrow(\mathcal{S}) \mid \uparrow$$
$$\mathcal{N} ::= \underline{0} \mid \mathbf{S}\mathcal{N}.$$

Figure: Terms of $\lambda\upsilon$-calculus.

$$\begin{aligned}
(\lambda a)b &\to a[b/] && \text{(Beta)}\\
(ab)[s] &\to a[s](b[s]) && \text{(App)}\\
(\lambda a)[s] &\to \lambda(a[\Uparrow(s)]) && \text{(Lambda)}\\
\underline{0}[a/] &\to a && \text{(FVar)}\\
(\mathbf{S}\underline{n})[a/] &\to \underline{n} && \text{(RVar)}\\
\underline{0}[\Uparrow(s)] &\to \underline{0} && \text{(FVarLift)}\\
(\mathbf{S}\underline{n})[\Uparrow(s)] &\to \underline{n}[s][\uparrow] && \text{(RVarLift)}\\
\underline{n}[\uparrow] &\to \mathbf{S}\underline{n}. && \text{(VarShift)}
\end{aligned}$$

Figure: Rewriting rules.

$\lambda\upsilon$-calculus (ii)

Consider the term $K = \lambda x.\lambda y.x$; denoted as $\lambda\lambda\underline{1}$. Certainly, $Kab \to_\beta a$ for each term a (in one step). Note however, that with explicit substitutions we have

$$\begin{aligned}(\lambda\lambda\underline{1})a &\to (\lambda\underline{1})[a/] \\ &\to \lambda(\underline{1}[\Uparrow(a/)]) \\ &\to \lambda\,(\underline{0}[a/][\uparrow]) \\ &\to \lambda\,(a[\uparrow])\,.\end{aligned}$$

The final shift operator guarantees that (potential) free indices are aptly incremented so to avoid potential variable captures. If a is closed, then $a[\uparrow]$ resolves simply to a, as intended.

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Enumerative results

Proposition

Let $T(z)$ and $S(z)$ denote the generating functions corresponding to $\lambda\upsilon$-terms and substitutions, respectively. Then,

$$T(z) = \frac{1-\sqrt{1-4z}}{2z} - 1 \quad \text{whereas} \quad S(z) = \frac{1-\sqrt{1-4z}}{2z}\left(\frac{z}{1-z}\right).$$

In consequence

$$[z^n]T(z) = \begin{cases} 0, & \text{for } n = 0 \\ \frac{1}{n+1}\binom{2n}{n}, & \text{otherwise} \end{cases}$$

$$[z^n]S(z) = \begin{cases} 0, & \text{for } n = 0 \\ \sum_{k=0}^{n-1} \frac{1}{k+1}\binom{2k}{k} & \text{otherwise.} \end{cases}$$

Enumerative results – explicit bijection

$$\varphi\left(\begin{array}{ccc} \bullet & & \\ & \diagdown & \\ & & R \end{array}\right) = \lambda\varphi(R)$$

$$\varphi\left(\begin{array}{ccc} & & \bullet \\ & \diagup & \\ L & & \end{array}\right) = \mathbf{S}\underline{\mathrm{n}}$$

$$\textbf{when} \quad \varphi(L) = \underline{\mathrm{n}}$$

$$\varphi\left(\begin{array}{ccc} & & \bullet \\ & \diagup & \\ L & & \end{array}\right) = a[\Uparrow^{n+1}(s)]$$

$$\textbf{when} \quad \varphi(L) = a[\Uparrow^{n}(s)]$$

$$\varphi(\bullet) = \underline{0}$$

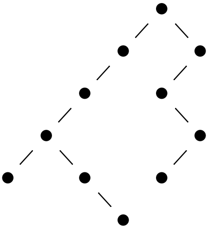
$$\varphi\left(\begin{array}{ccc} & & \bullet \\ & \diagup & \\ \bullet & & \\ & \diagdown & \\ & & R \end{array}\right) = \varphi(R)[\uparrow]$$

$$\varphi\left(\begin{array}{ccccc} & & & & \bullet \\ & & & \diagup & \\ & & \bullet & & \\ & \diagup & & \diagdown & \\ L & & & & R \end{array}\right) = \varphi(L)[\varphi(R)/]$$

$$\varphi\left(\begin{array}{ccccc} & & \bullet & & \\ & \diagup & & \diagdown & \\ L & & & & R \end{array}\right) = \varphi(L)\,\varphi(R)$$

Enumerative results – correspondence example

For instance, $\underline{0}[\Uparrow (\lambda \underline{0}/)]\ \underline{1}[\uparrow]$ corresponds to



Statistical properties

Strict substitution forms

A $\lambda\upsilon$-term t is in *strict substitution form* if there exist two pure (i.e. without explicit substitutions) terms a, b and a sequence $t_1, \ldots, t_n$ of $\lambda\upsilon$-terms such that

$$a[b/] \to t_1 \to \cdots \to t_n = t$$

and none of the above reductions is (Beta).
Otherwise, t is said to be in *lazy substitution form*.

Theorem

Asymptotically almost all $\lambda\upsilon$-terms are in *lazy substitution form*.

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Proof.

Idea: Consider the class of terms without *nested closures*. $\square$

Statistical properties (ii)

Substitution suspension

Let s be an substitution and t be a $\lambda\upsilon$-term. Then, s, all its subterms, and all the constructors it contains are said to be *suspended in t* if t contains a subterm in form of $[s]$; in other words, when s occurs under a closure in t.

Theorem

Let X_n be a random variable denoting the number of constructors not suspended under a closure in a random $\lambda\upsilon$-term of size n. Then, the expectation $\mathbb{E}(X_n)$ satisfies

$$\mathbb{E}(X_n) \xrightarrow[n\to\infty]{} \frac{316}{3} \approx 105.33.$$

Statistical properties (iii)

Proposition

Let R be a redex in the $\lambda\upsilon$-calculus and X_n be the corresponding random variable denoting the number of occurrences of R in a random term of size n. Then, after standardisation, X_n admits a limiting Gaussian law.

In particular, we obtain:

Redex	(limit) mean	(limit) variance
(Beta)	$0.046875n$	$0.037354n$
(App)	$0.031250n$	$0.021973n$
(Lambda)	$0.031250n$	$0.025879n$
(VarShift)	$0.015625n$	$0.013916n$
(FVar)	$0.011719n$	$0.011124n$
(FVarLift)	$0.007812n$	$0.007355n$
(RVar)	$0.003906n$	$0.003799n$
(RVarLift)	$0.002604n$	$0.002557n$

Conclusions

- Standard analytic methods provide insight into the structure of random $\lambda\upsilon$-terms, in particular *substitution resolution*;

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- Standard analytic methods provide insight into the structure of random $\lambda\upsilon$-terms, in particular *substitution resolution*;
- Typical computations are in a strong sense *lazy*. Substitutions are evaluated non-strictly. In fact, almost all of the computation content is *suspended*;
- Substitution primitives have, on average, an uneven contribution in typical terms. We can exploit that in *micro optimisations* of abstract machines.

Future work

- What is the cost distribution of substitution resolution, given a random term of size n? What primitives are its main contributors?

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- Does normal-order reduction (think of non-strict functional programming languages) has a similar, average-case "distribution shape" as the one for combinatory logic?

Thank you

Thank you for your attention!
(Strict questions and lazy answers)