On Domain Theory over Girard Quantales

A QUESTION FOR THE AUDIENCE

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Quantale semantics

David N. Yetter (1990)

Quantales and (Noncommutative) Linear Logic,
The Journal of Symbolic Logic, **55**(1), pp. 41–64.

THEOREM. The (Girard) quantale semantics for commutative LL is sound and complete with respect to the commutative linear sequent calculus:

$$\vdash A_1, A_2, ..., A_n \text{ iff } \mathbf{I} \leqslant |A_1 \parr A_2 \parr ... \parr A_n|.$$

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- Informally: $\wedge, \vee, \otimes, \invamp, \multimap, \bigvee, \bigwedge, 0, 1, \mathbf{I}, \bot, \neg, !, ?$

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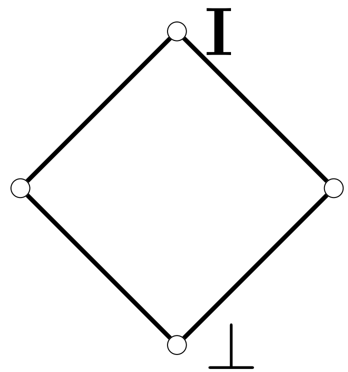
- Recall: $\vdash A_1, A_2, ..., A_n$ iff $\mathbf{I} \leqslant |A_1 \invamp A_2 \invamp ... \invamp A_n|$
  iff $\mathbf{I} \leqslant |A_1| \invamp |A_2| \invamp ... \invamp |A_n|$

Examples

- Every complete Boolean algebra is a Girard quantale with $\otimes = \wedge$, e.g.:



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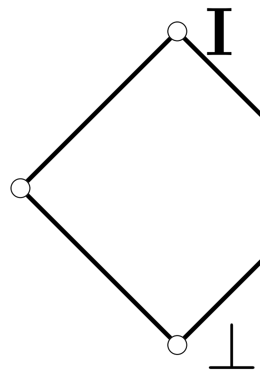

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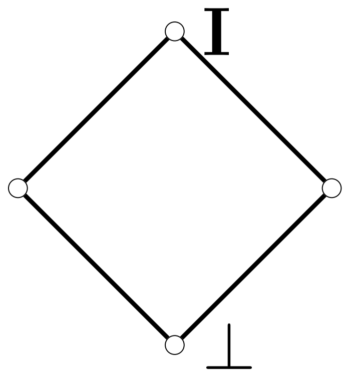

- The two-element lattice $\mathbf{2} = \{\mathbf{I}, \perp\}$ with $\otimes = \wedge$
- The unit interval $([0,1], \geqslant)$ with $\mathbf{I} = 0$, $\perp = 1$, $\otimes = +$

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5 I don't understand why this happens!

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- When $\mathcal{Q} = \mathbf{2}$, the above axioms represent poset axioms exactly.
- However, when $\mathcal{Q} = [0,1]$, the above axioms become metric axioms!

Coding domain theory – example 1

- For a subset $A \subseteq X$ of a poset $(X, \sqsubseteq)$, A is lower if

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- Def.: A is a lower if

$$\mathbf{I} \leqslant \bigwedge_x \bigwedge_y [(A(y) \otimes X(x,y)) \multimap A(x)].$$

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- For a subset $A \subseteq X$ of a poset $(X, \sqsubseteq)$, A is Scott-open if

$$\forall \phi \in \mathcal{I}X\ [\mathcal{S}\phi \in A \iff (\exists x \in \phi\ (x \in A))].$$

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- In $\mathbf{2}$ this means that a subset H has the property that $\mathcal{S}\phi \in H$ iff $\phi \subseteq H$, which is exactly the definition of a Scott-closed subset H!

...and in this way I can define much more: continuous maps, way-below relation, closure operators, compact subsets, abstract bases, powerdomains, etc. ...and prove all of the usual basic theorems of domain theory in more general setup.

WHY ?!

What I think is relevant

- Inside every Girard quantale $\mathcal{Q}$ lives a complete Boolean algebra:

$$\mathcal{B} = \{?!x \mid x \in \mathcal{Q}\}.$$

$$x \wedge_{\mathcal{B}} y := ?!(x \wedge y)$$
$$x \vee_{\mathcal{B}} y := x \vee y$$
$$x \Rightarrow_{\mathcal{B}} y := !x \multimap y$$
$$0_{\mathcal{B}} = \perp$$
$$1_{\mathcal{B}} = ?\mathbf{I}$$
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$$\bigwedge_{\mathcal{B}} S =?! \bigwedge S$$
$$\bigvee_{\mathcal{B}} S =? \bigvee \{!x \mid x \in S\}$$

- For $! = \texttt{ext}$, where $\texttt{ext}(x) := \mathbf{I}$ if $x = \mathbf{I}$, and $\texttt{ext}(x) = \perp$ otherwise, we get $\mathcal{B} = \{\perp, \mathbf{I}\} = \mathbf{2}$.

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- CAN I HOPE FOR a similar theorem concerning a translation of domain theory from classical to linear logic?

The end

Fin

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