Dark matter among and/or trees

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• $\operatorname{Var} = \{x_1, x_2, x_3, \dots\}$ - countable set of variables

• $\operatorname{Var}_k = \{x_1, \ldots, x_k\}$

Definition

 \mathcal{T}_k - set of and/or trees with k-variables is a minimal set such that:

x_i, ¬x_i ∈ T_k, for each x_i ∈ Var_k,
 (t₁ ∧ t₂), (t₁ ∨ t2), for each t₁, t₂ ∈ T_k
 T_k(n) - the set of elements of T_k of size n.

Observation

Every $t \in \mathcal{T}_k$ defines some function $I(t) : \{0,1\}^k \to \{0,1\}$.

$$(I:\mathcal{T}_k \to \{0,1\}^{\{0,1\}^k})$$

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• $(t_1 \wedge t_2), (t_1 \vee t_2)$, for each $t_1, t_2 \in \mathcal{T}_k$

 $\mathcal{T}_k(n)$ - the set of elements of \mathcal{T}_k of size n.

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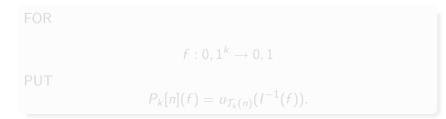
 $u_{T_k(n)}$ - uniform probability distribution on the (finite) set $\mathcal{T}_k(n)$. FOR $f:0,1^k o 0,1$ PUT $P_k[n](f)=u_{\mathcal{T}_k(n)}(I^{-1}(f)).$

 $P_k[n]$ is a probability distribution on $\{0,1\}^{\{0,1\}^k}$.

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Theorem (Lefmann, Savicki 1997)

There exists distribution P_k on the set $\{0,1\}^{\{0,1\}^k}$ such that:

$$P_k[n](f) \to_{n \to \infty} P_k(f)$$

for every $f \in \{0,1\}^{\{0,1\}^k}$.

Equivalently

$$P_k(f) = \lim_{n \to \infty} \frac{Card((I^{-1}(f) \cap \mathcal{T}_k(n)))}{Card(\mathcal{T}_k(n))}$$

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Non constant f.

Upper bound - B. Chauvin, P. Flajolet, D. Gardy, and B. Gittenberger 2004

$$P_k(f) \le (1 + O(1/k))exp(-c\frac{L(f)}{k^2})$$

for some constant c > 0.

Lower bound - Lefmann, Savicki 1997

$$\frac{1}{4}(8k)^{-L(f)-1} \le P_k(f)$$

Improved lower bound - D. Gady, A. Woods 2005

$$\frac{4m(f)}{(16k)^{L(f)+1}} \le P_k(f)$$

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Focus on low complexities.

f - non-constant function. There exists positive constant B_f such that

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Corollary

Lower bound is quite good for the functions with very low complexity.

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What next? Big complexity ?

Uniform counting result

A fraction tending to 1 (as $k \to \infty$) of boolean functions in k variables have tree complexity at least $2^k / \log k$.

Guess

The set of functions with high complexity should have a big measure $P_k(_)$.

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But... Why the result for low complexities hold?

Definition (Negative pattern

$$N \to N \lor N | \Box \land N | \bullet$$

Definition (Positive pattern)

$$P \to \Box \lor P \middle| P \land P \middle| \bullet$$

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Distribution of numbers of positive leaves

Prob(tree of size *n* has *m* positive leaves)
$$\rightarrow_{n \rightarrow \infty} C_{m-1}(2m-1)(\frac{2}{\alpha})^m$$

Corollary

Most of trees have small number od positive leaves.

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In most of and/or trees the variables in the positive leaves are all different.

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the table for I(t) contains hypercube of dimension n - m filled with 1.

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Big costant hypercubes are quite rare among boolean functions.

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The family

F_m - all the functions definable by trees with at most m positive leaves.

 $G_k = F_{\log k} \cap \{0,1\}^{\{0,1\}^k}$

THEN

 $P(G_{\log k}) \rightarrow_{k \rightarrow \infty} 1$

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There exist functions whose complexity is neither small nor heigh which are dominant for the distribution *P*.

What functions are they?

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