Algebraic complexity and computational geometry

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Outline

- Models of computation, lower bounds, complexity of Element Distintness.
- $\Omega(n \log n)$ lower bound for computing the diameter of a 3D convex polytope.
- Hopcroft's problem and the diameter in higher dimension.

Models of computation, complexity of Element Distintness

Example of geometric problems

- Element Distinctness
  - Input: $x_1, \ldots, x_n \in \mathbb{R}$
  - Decide if the x_i are distinct
- Diameter in $\mathbb{R}^d$
  - Input: $p_1, \ldots, p_n \in \mathbb{R}^d$
  - Compute the euclidean diameter of $\{p_1, \ldots, p_n\}$
  - *or* Decide if the diameter is smaller than 1
- Hopcroft's Problem
  - Input: points $p_1, \ldots, p_n$ and lines $\ell_1, \ldots, \ell_n$ in $\mathbb{R}^2$
  - Decide if $\exists i, j$ such that $p_i \in \ell_j$

Model of computation: Real-RAM

- Real Random Access Machine.
- Each registers stores a *real* number.
- Access to registers in unit time.
- Arithmetic operation $(+, -, \times, /)$ in unit time.

Element Distinctness

- $O(n \log n)$ upper bound
  - Sort the input points $x_1, \ldots, x_n$
  - $x_{\sigma(1)} \leqslant \ldots \leqslant x_{\sigma(n)}$ is obtained in time $O(n \log n)$
  - Check if there exists i such that $x_{\sigma(i)} = x_{\sigma(i+1)}$
- Lower bound?

Structure of Element Distinctness

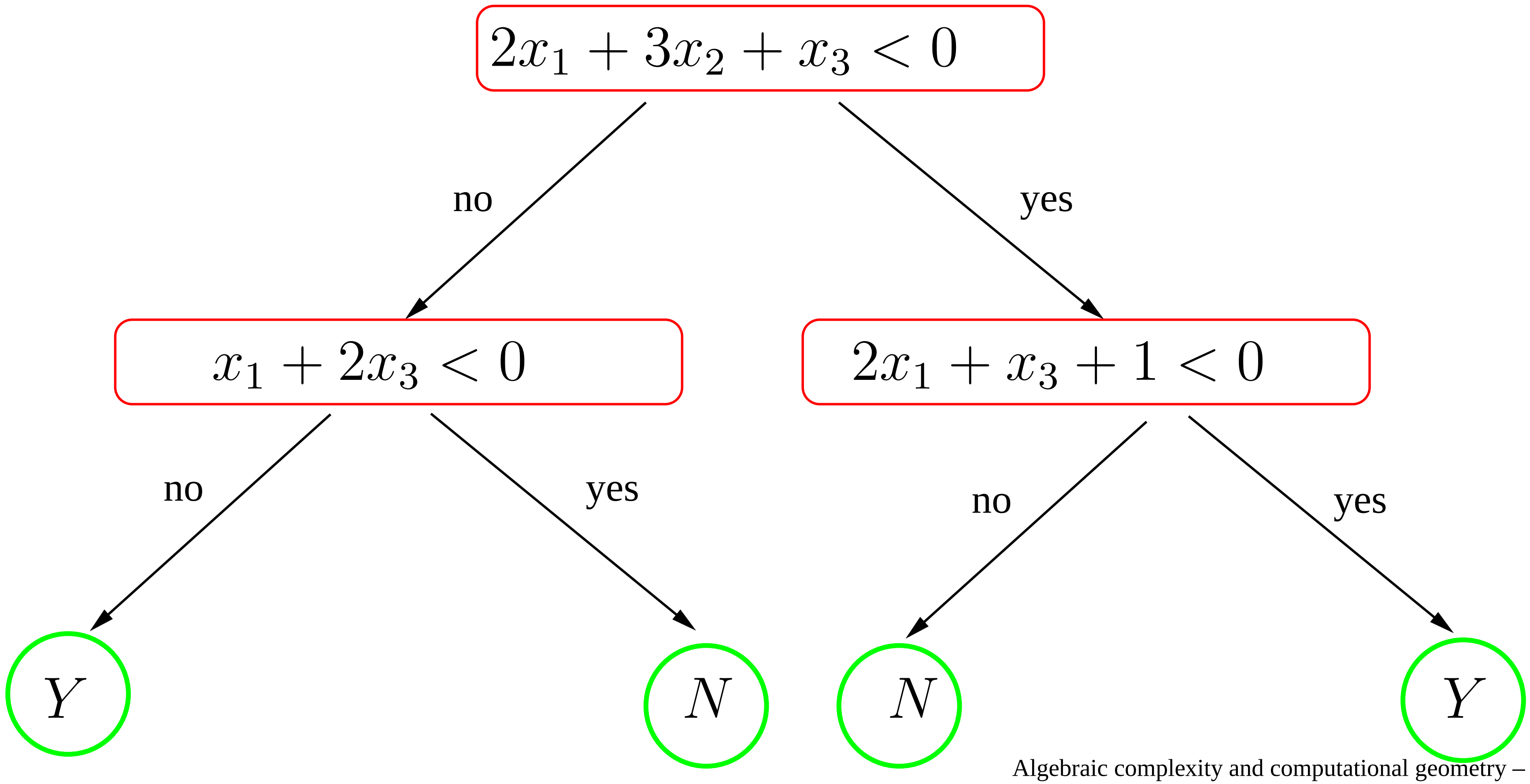
- Inputs of size n: we have to decide the complement of the union of all hyperplanes of equations $x_i = x_j$ in $\mathbb{R}^n$
- Number of connected components:
  - $2^{\binom{n}{2}}$ sign conditions ($x_i < x_j$ or $x_i > x_j$ for each $i < j$)
  - Some sign conditions are not compatible, eg $x_1 < x_2$, $x_2 < x_3$ and $x_3 < x_2$
  - Compatible sign conditions correspond to permutations σ:

$$x_{\sigma(1)} < x_{\sigma(2)} < \ldots < x_{\sigma(n)}.$$

  - There are exactly $n!$ cells in the arrangement

Lower bound in a weak model

- Real-RAM with $+$ and scalar multiplications $\cdot\lambda : x \mapsto \lambda x$
- Algorithm in time t can be unfolded into a family (T_n) of *Linear decision trees* of depth $t(n)$



Linear decision trees

- Dobkin et Lipton bound:
  - If S decided by a tree of depth d, then

$$\#CC(S) \leqslant 2^d$$

  - Application to Element Distinctness: depth d_n for inputs of size n is bounded by

$$n \log n \approx \log(n!) \leqslant d_n$$

  - As a consequence, the complexity of Element Disctintness is $\Theta(n \log n)$ in the linear real-RAM model (and in the Linear decision tree model)

Linear decision trees

- Linear decision tree is a powerful model
  - Subset Sum: given $(x_1, \ldots, x_n)$ decide if

$$\exists I \subseteq \{1, \ldots, n\}, \sum_{i \in I} x_i = 1$$

  - Can be solved by poly-depth linear decision trees
  - Subset Sum is NP-complete over $(\mathbb{R}, +, <)$
  - Can be solved by poly-time real (additive) algorithm iff P=NP (classical, non uniform)

Decision trees with arbitrary polynomials

- Too powerful to obtain any lower bound on Element Distinctness
  - Only one test:

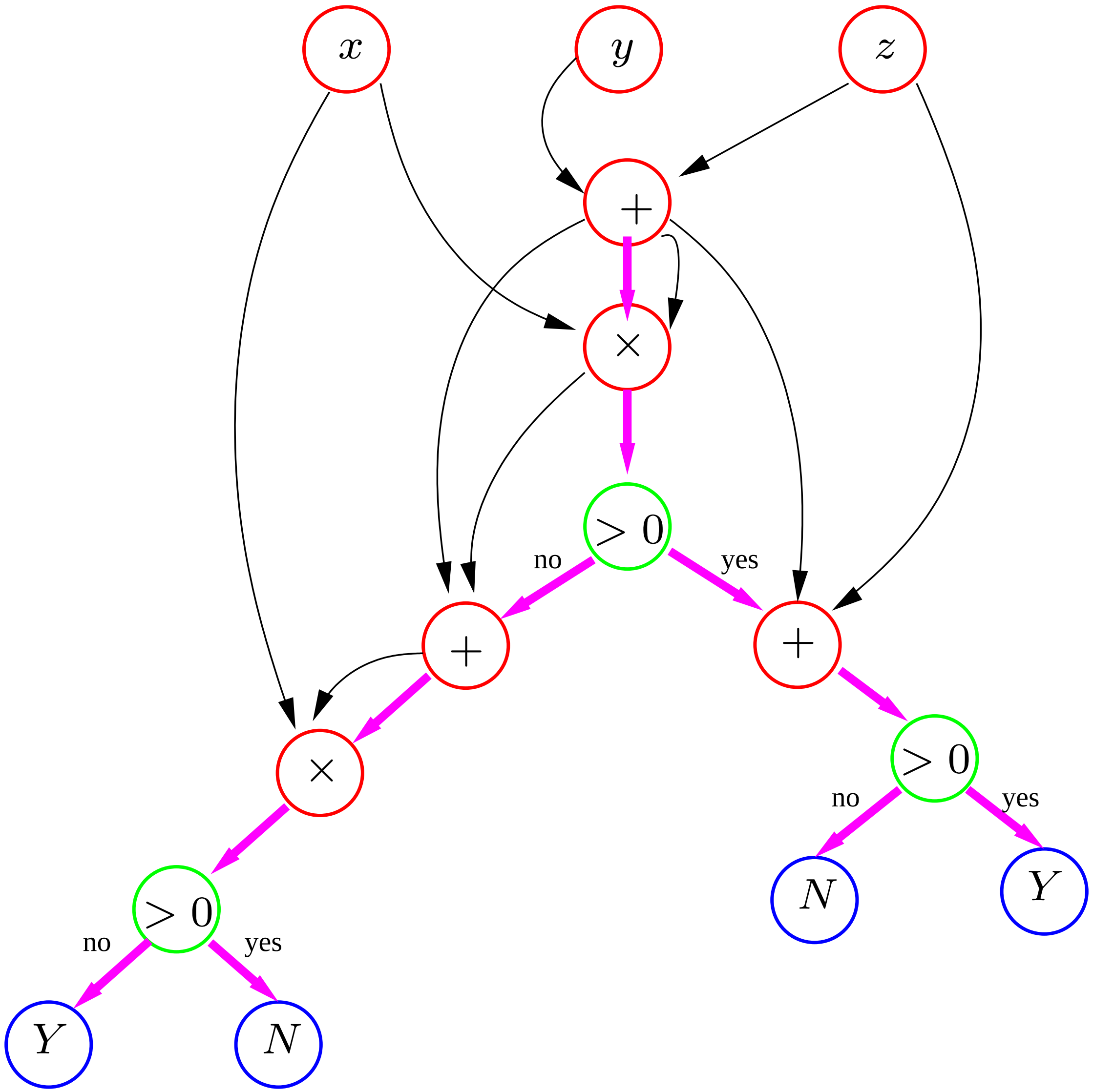
$$\prod_{i<j}(x_i - x_j) = 0$$

- The model has to take into account the complexity of computing the tests

Algebraic computation tree

- Input: $\bar{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.
- Output: YES or NO
- It is a tree with 3 types of nodes
  - Computation nodes:
    - a real constant,
    - some input number x_i, or
    - an operation $\{+, -, \times, /\}$ performed on ancestors of the current node.
  - Branching nodes: compares with 0 the value obtained at a computation node that is an ancestor of the current node.
  - Leaves: YES or NO

Algebraic computation tree: example



Algebraic computation tree (ACT)

- We say that an ACT *decides* $S \subset \mathbb{R}^n$ if
  - $\forall (x_1, \ldots, x_n) \in S$, it reaches a leaf labeled YES, and
  - $\forall (x_1, \ldots, x_n) \notin S$, it reaches a leaf labeled NO.
- The ACT model is stronger than the real–RAM model.
- To get a lower bound on the worst-case running time of a real-RAM that decides S, it suffices to have a lower bound on the *depth* of all the ACTs that decide S

Theorem (Ben-Or)**.** *Any ACT that decides S has depth*

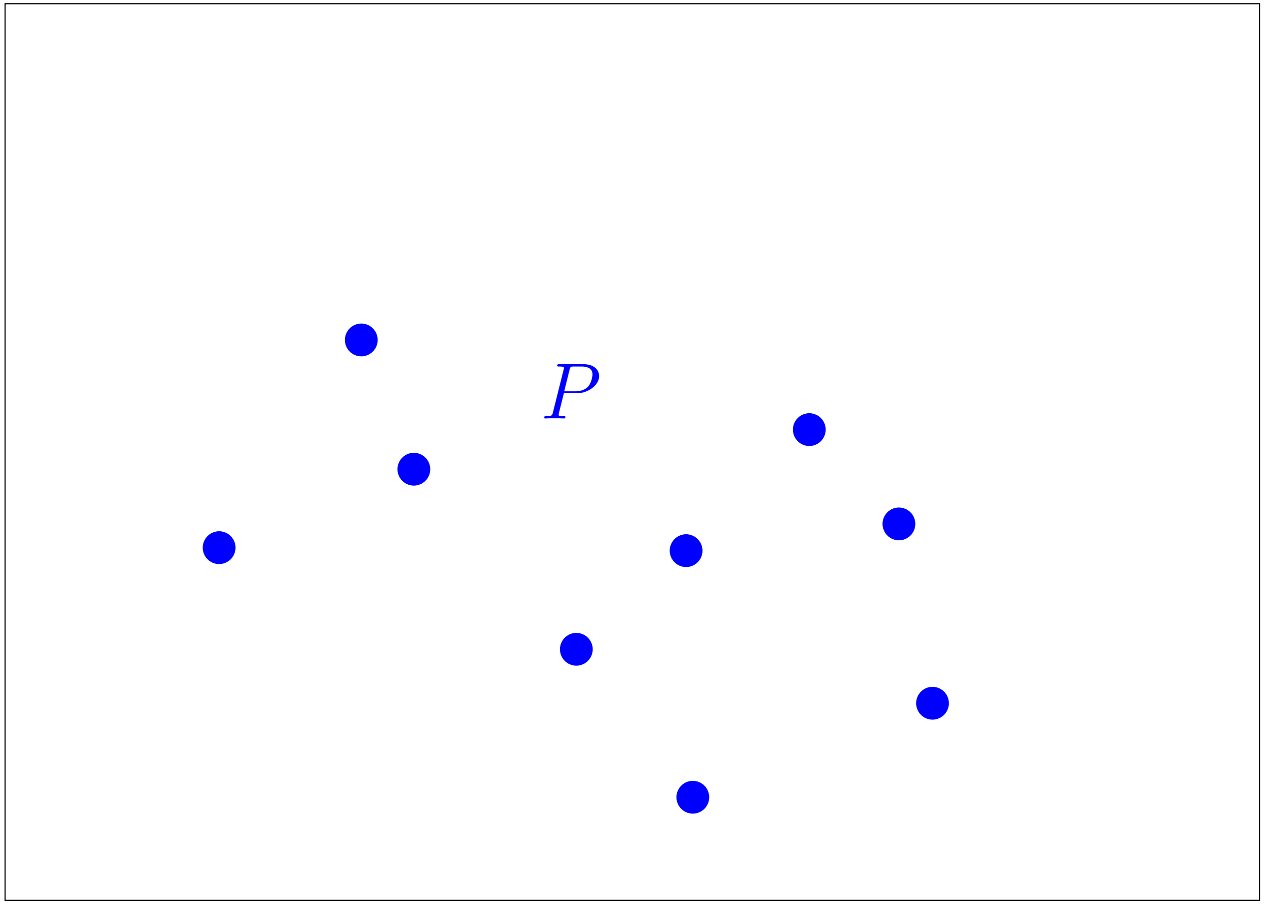
$$\Omega(\log \#CC(S)).$$

Complexity of Element Distinctness

- $\Theta(n \log n)$ in both real-RAM and ACT models

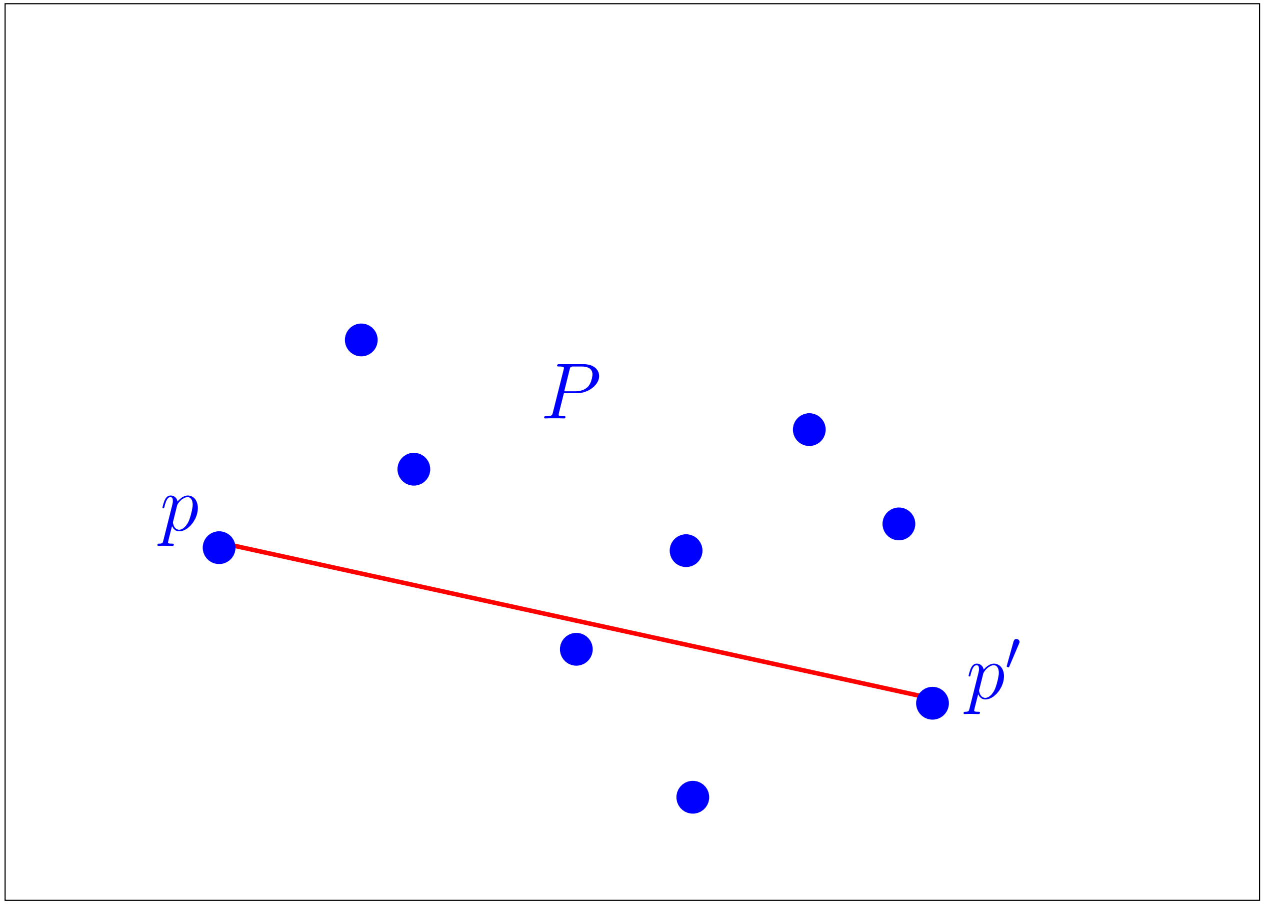
Lower bound for 3D convex polytopes

The diameter problem



- INPUT: a set P of n points in $\mathbb{R}^d$.
- OUTPUT: $\mathrm{diam}(P) := \max\{\mathrm{d}(x, y) \mid x, y \in P\}$.

The diameter problem

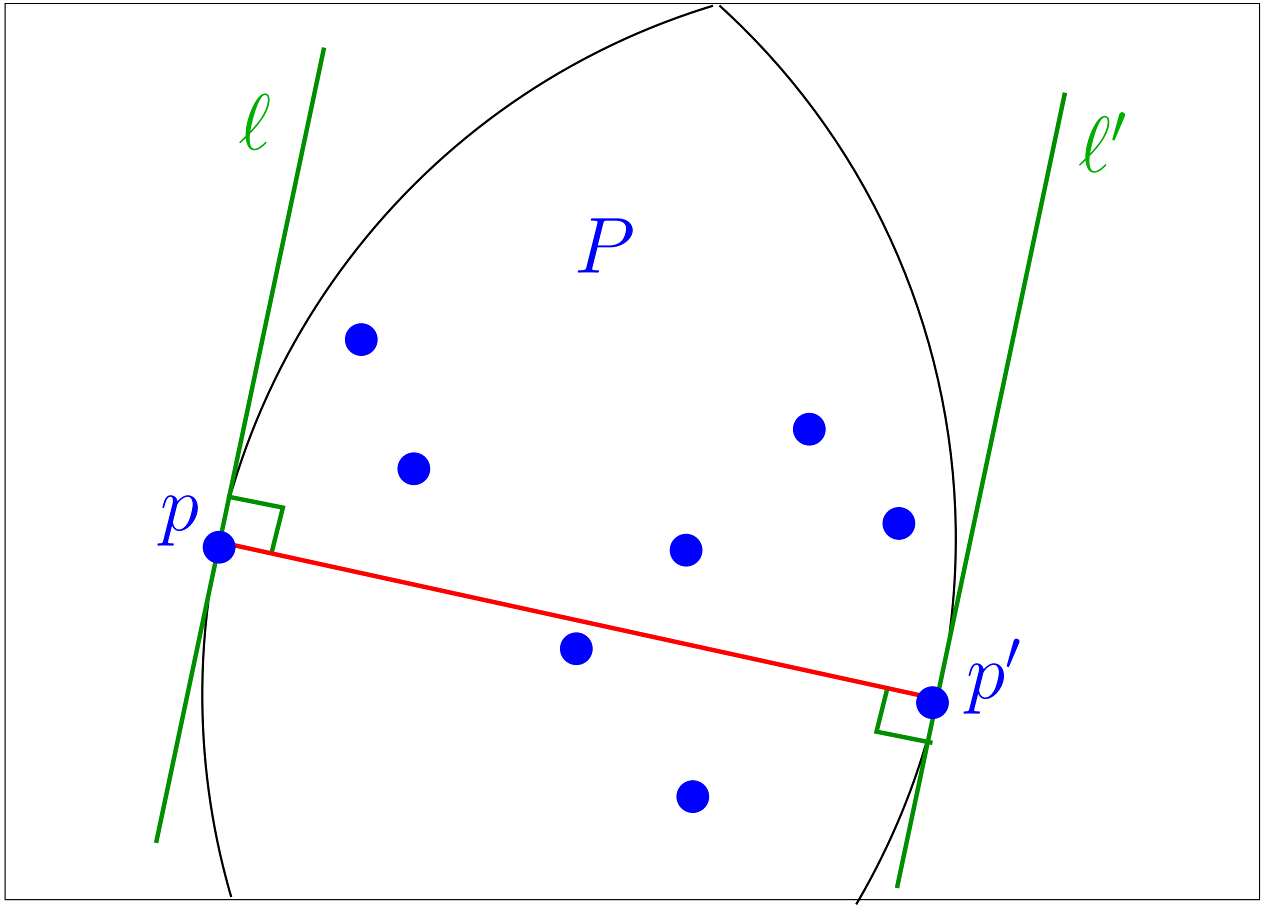


- $\mathrm{diam}(P) = \mathrm{d}(p, p')$.

Decision problem

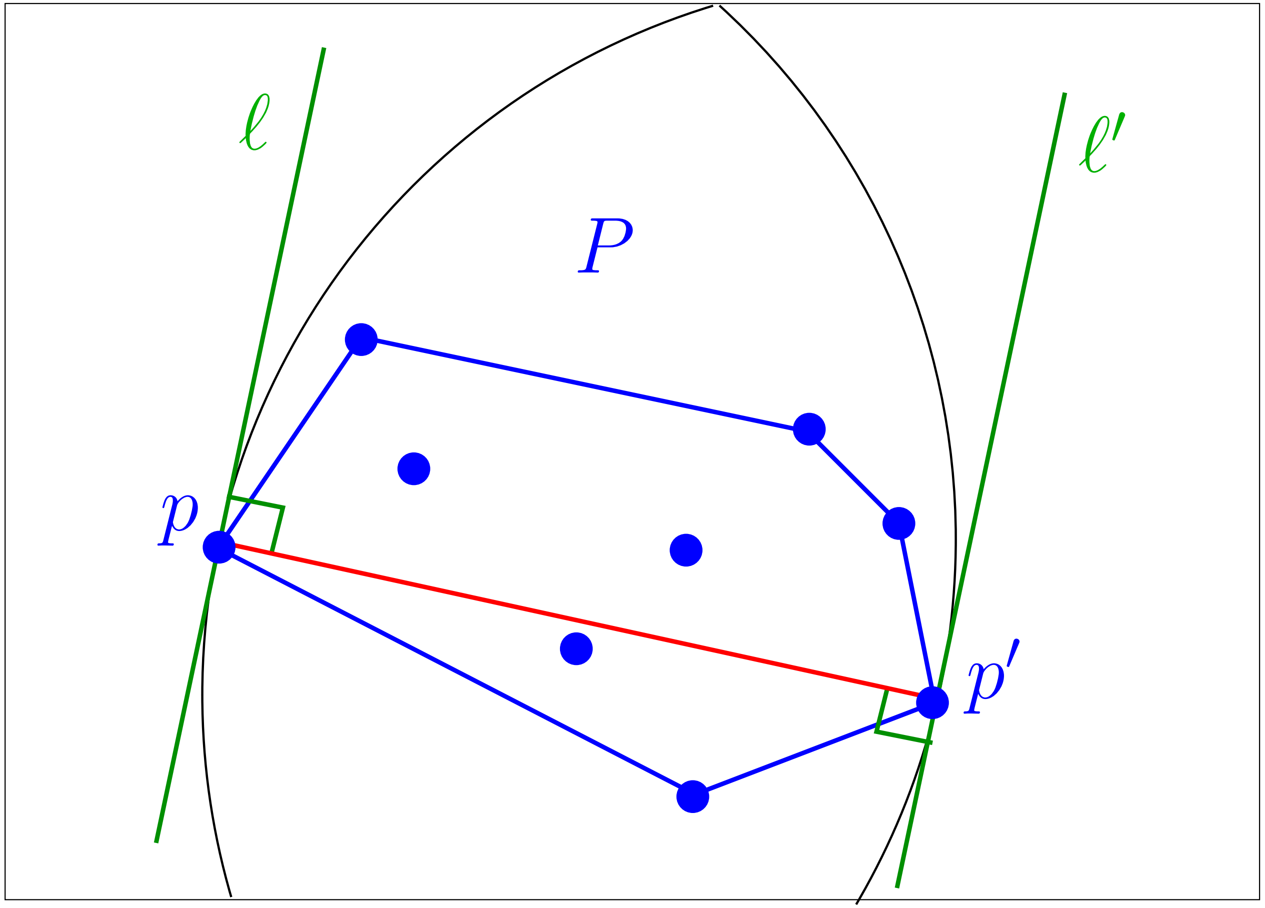
- We will give lower bounds for the *decision problem* associated with the diameter problem.
- INPUT: a set P of n points in $\mathbb{R}^d$.
- OUTPUT:
  - YES if $\operatorname{diam}(P) < 1$
  - NO if $\operatorname{diam}(P) \geqslant 1$

The diameter problem



- P lies between two parallel hyperplanes through p and p'. We say that (p, p') is an *antipodal pair*.

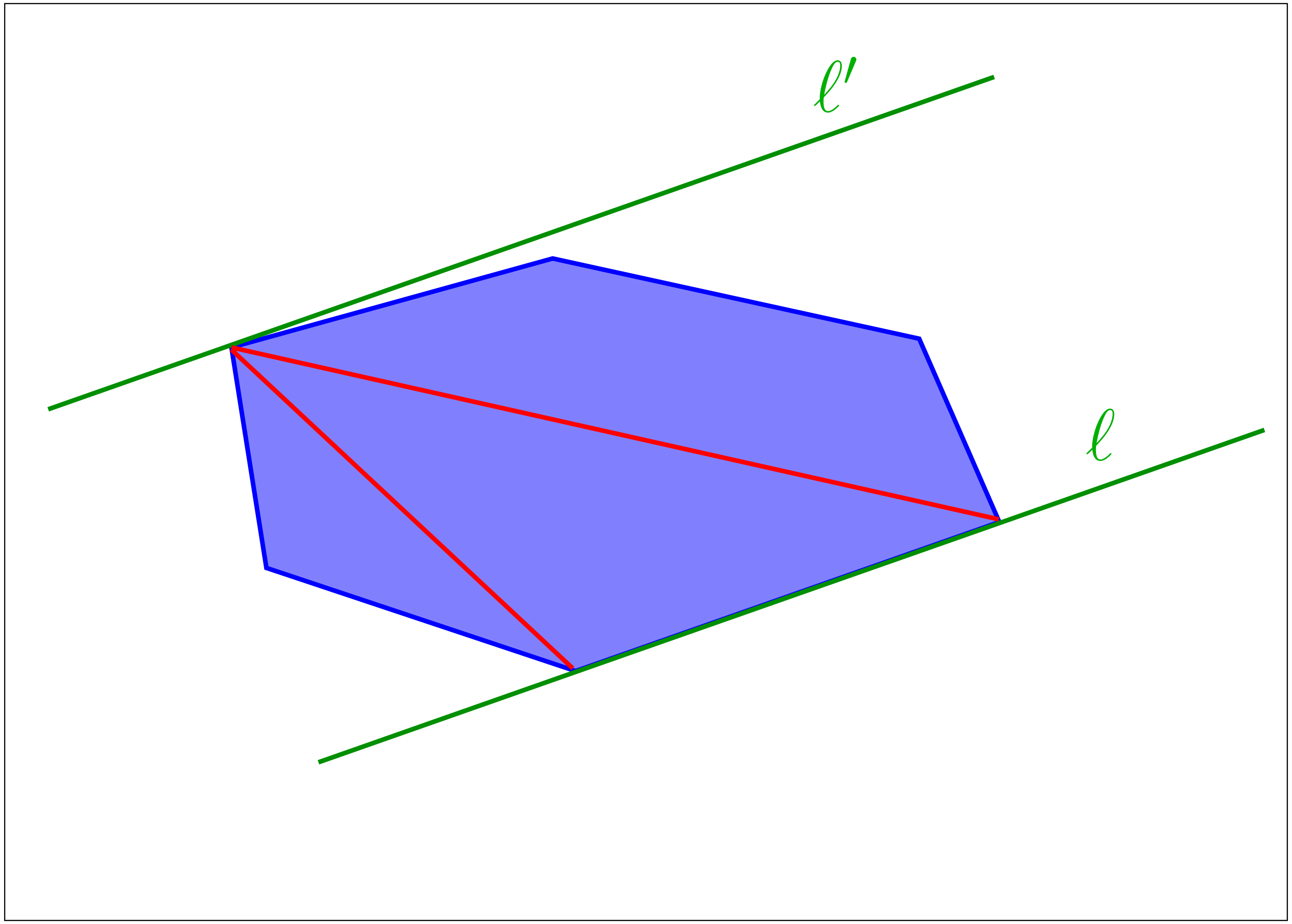
The diameter problem



- Any antipodal pair (and therefore any diametral pair) lies on the convex hull $\mathrm{CH}(P)$ of P.

Finding the antipodal pairs

- The rotating calipers technique.



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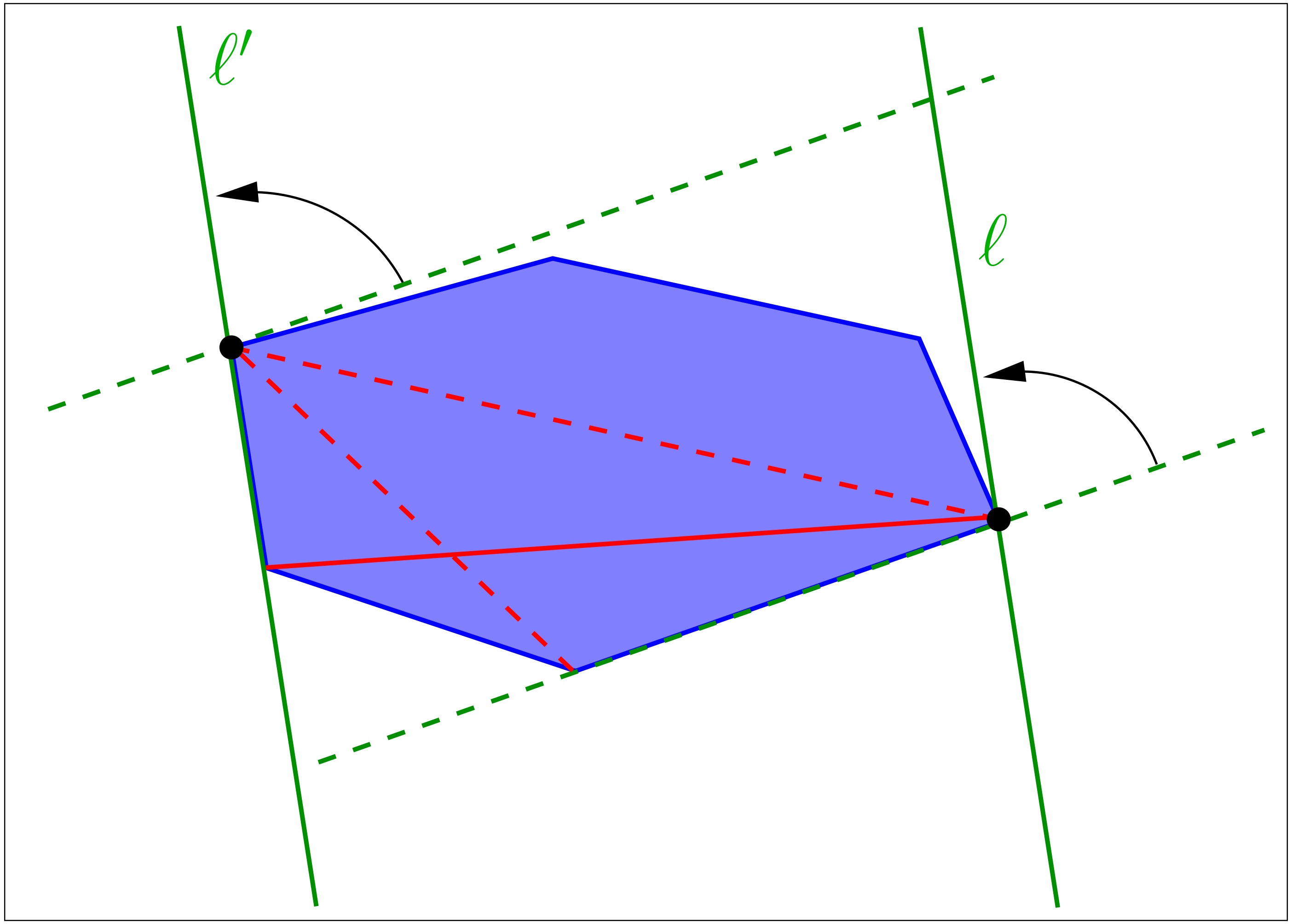

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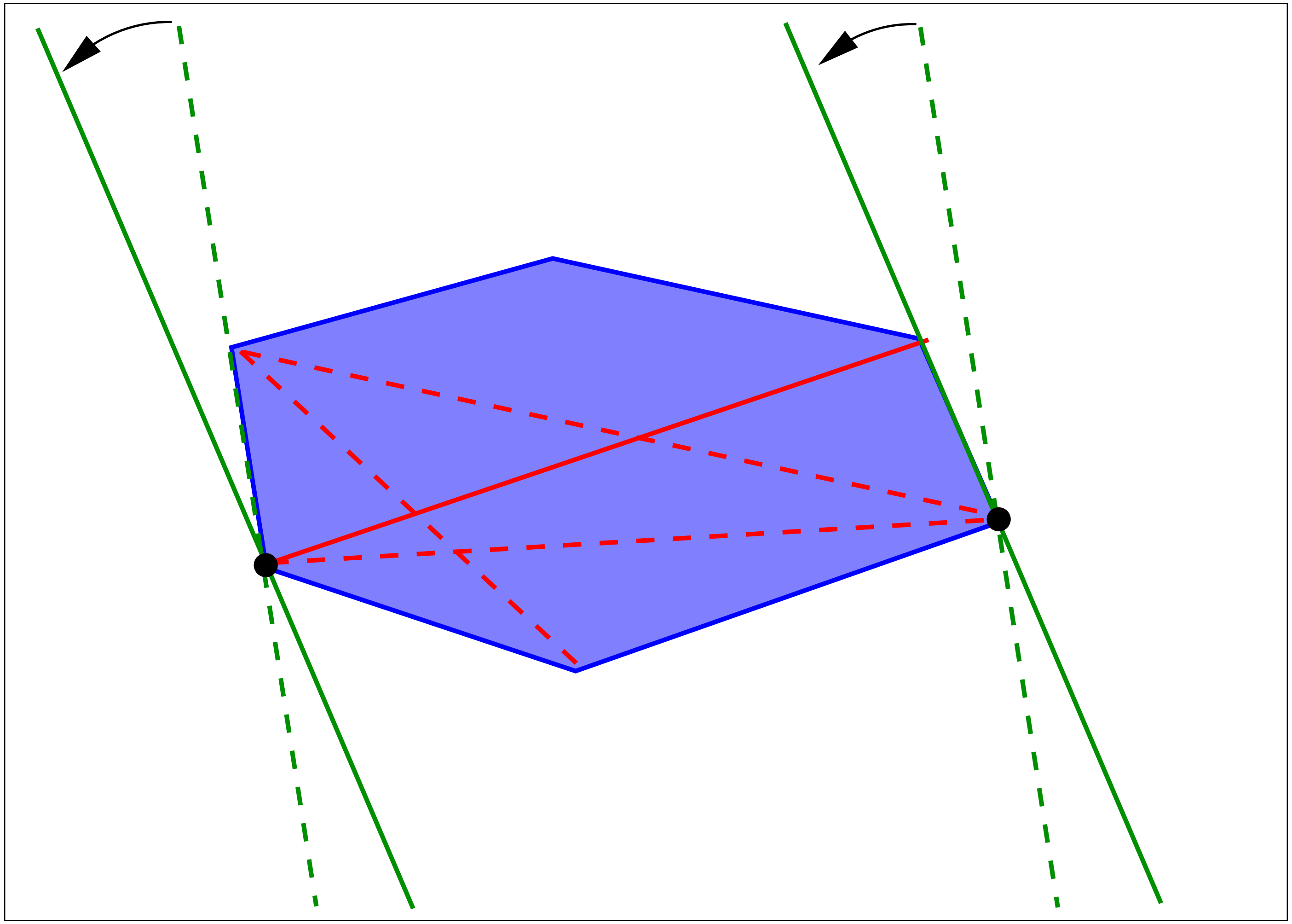

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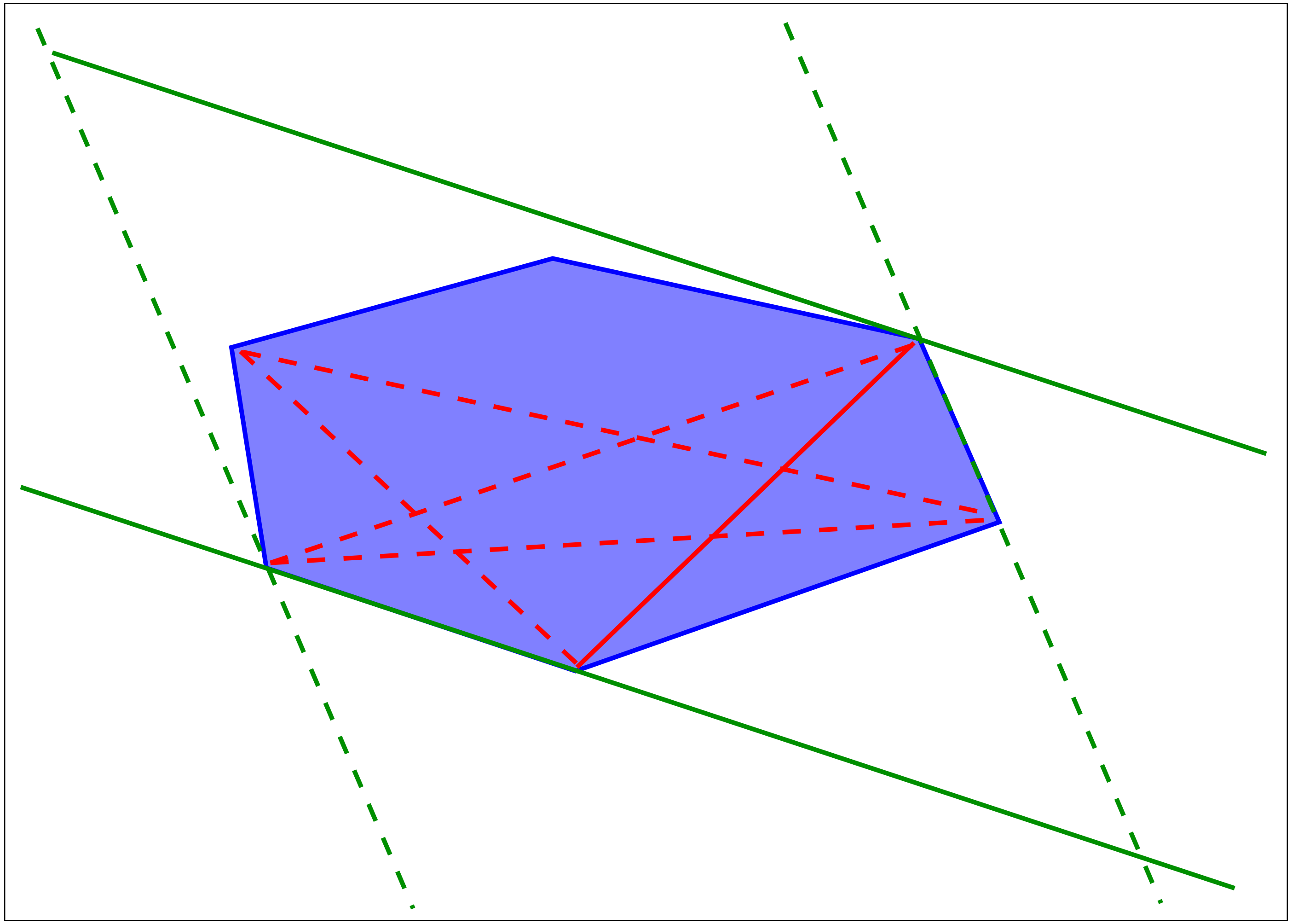

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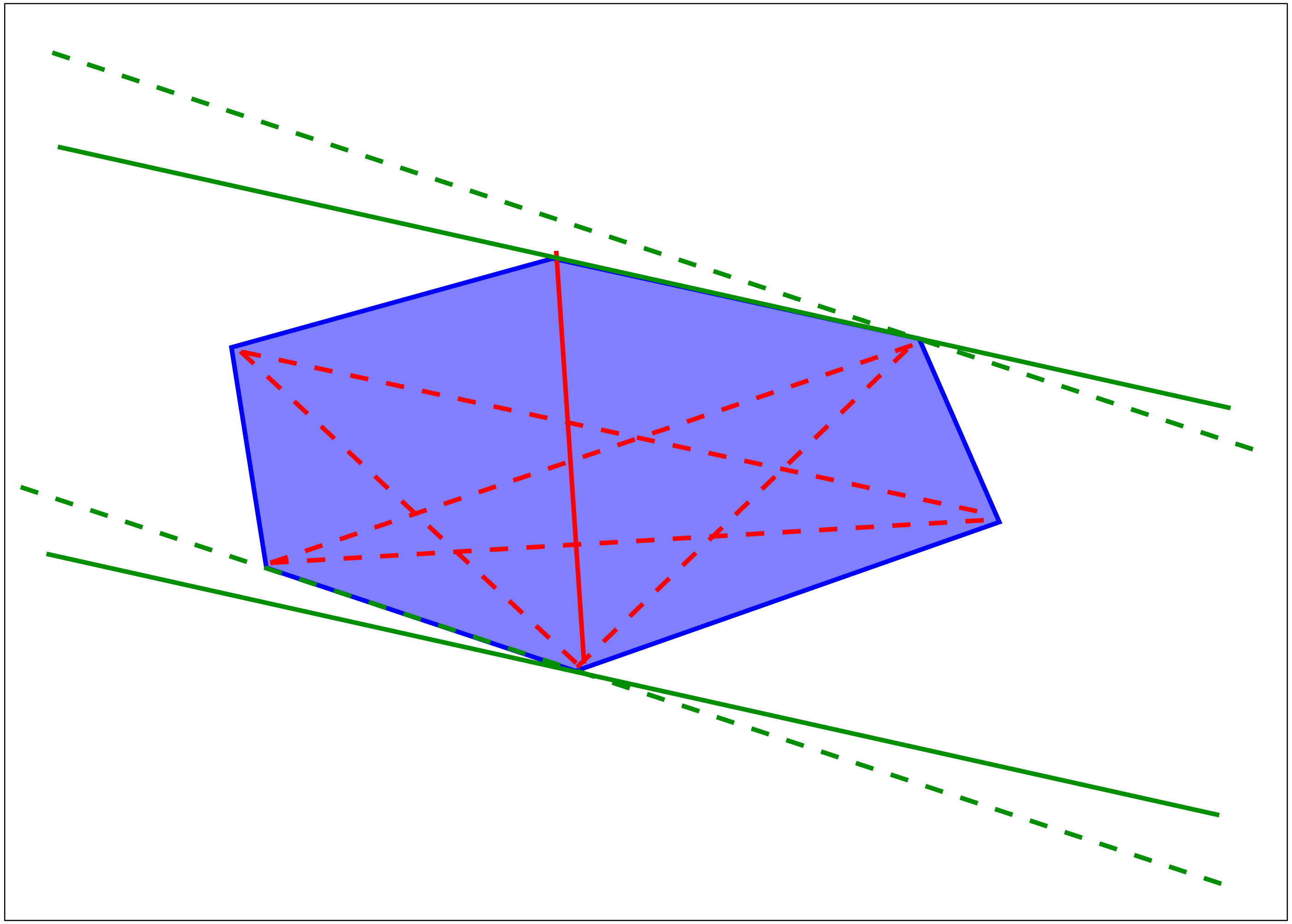


Computing the diameter of a 2D-point set

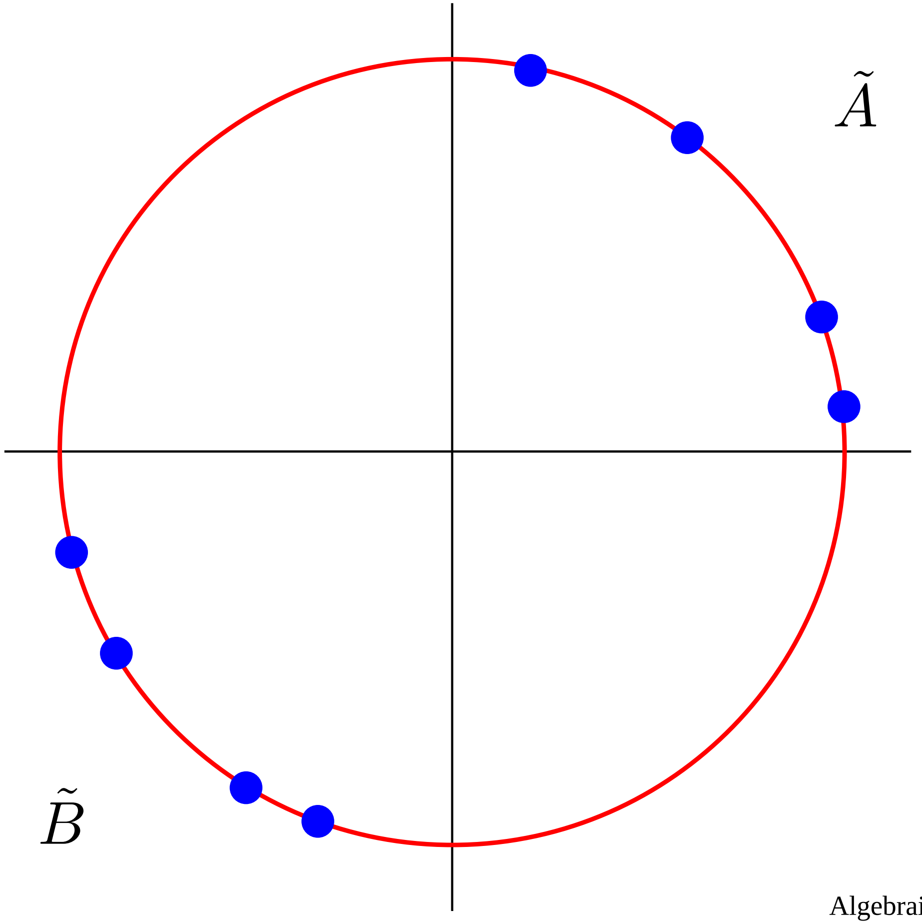
- Compute the convex hull $\mathrm{CH}(P)$ of P.
  - $O(n \log n)$ time.
- Find all the antipodal pairs on $\mathrm{CH}(P)$.
  - There are at most n such pairs in non–degenerate cases.
  - $O(n)$ time using the rotating calipers technique.
- Find the diametral pairs among the antipodal pairs.
  - $O(n)$ time by brute force.
- Conclusion:
  - The diameter of a 2D-point set can be found in $O(n \log n)$ time
  - The diameter of a convex polygon can be found in $O(n)$ time.

Diameter in $\mathbb{R}^3$ and higher dimensions

- Randomized $O(n \log n)$ time algorithm in $\mathbb{R}^3$ (Clarkson and Shor, 1988).
  - Randomized incremental construction of an intersection of balls and decimation.
- Deterministic $O(n \log n)$ time algorithm in $\mathbb{R}^3$ (E. Ramos, 2000).
- In $\mathbb{R}^d$, algorithm in $n^{2-2/(\lceil d/2 \rceil+1)} \log^{O(1)} n$ (Matoušek and Schwartzkopf, 1995).

Lower bound on the diameter

- $\Omega(n \log n)$ lower bound in $\mathbb{R}^2$.
  - Reduction from Set Disjointness.
    Given $A, B \subset \mathbb{R}$, decide if $A \cap B = \emptyset$.



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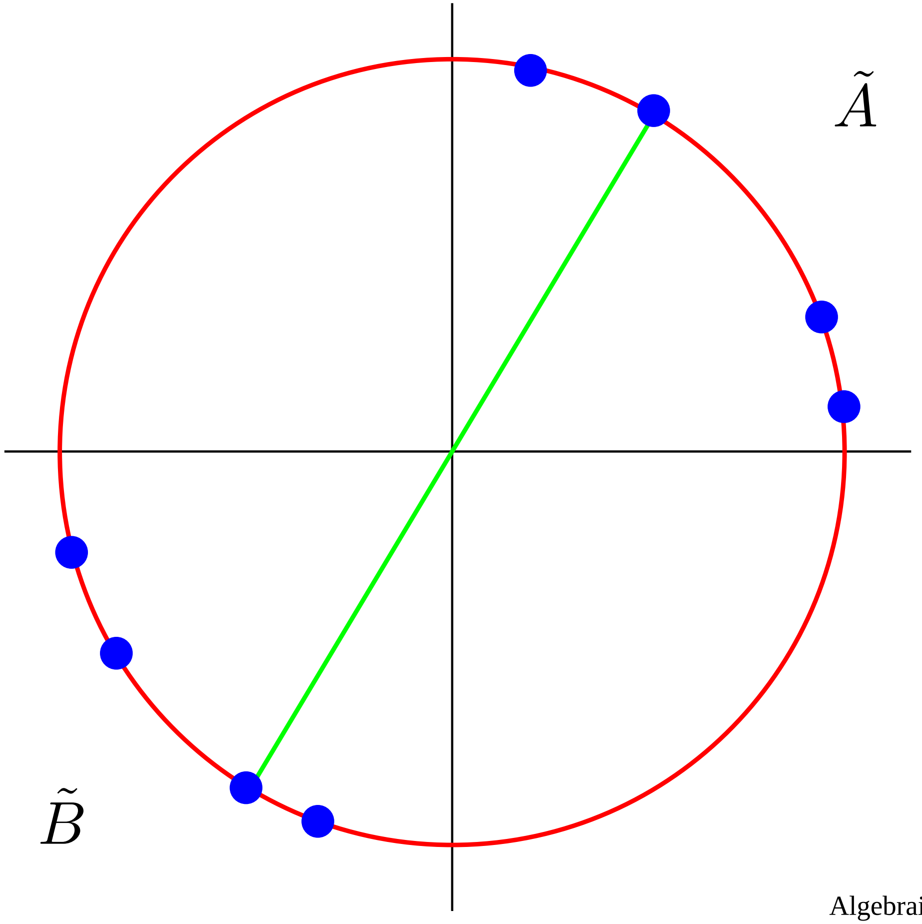


Diameter of a polytope

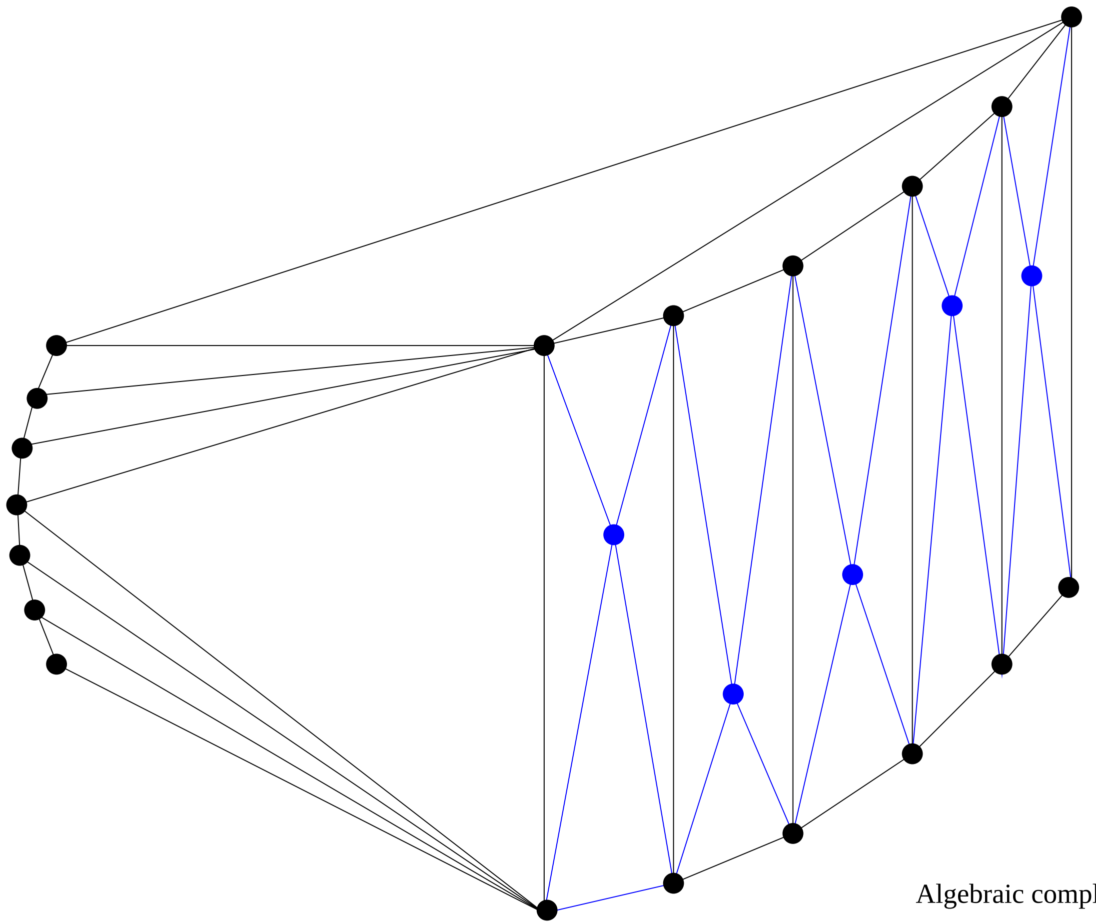
- The diameter of a convex polygon in $\mathbb{R}^2$ can be found in $O(n)$ time.
- Can we compute the diameter of a convex $3D$-polytope in linear time?

Problem statement

- We are given a convex 3-polytope P with n vertices.
- P is given by the coordinates of its vertices and its *combinatorial structure*:
  - All the inclusion relations between its vertices, edges and faces.
  - The cyclic ordering of the edges of each face.
- Remark: the combinatorial structure has size $O(n)$.
- Problem: we want to decide whether $\mathrm{diam}(P) < 1$.
- We show an $\Omega(n \log n)$ lower bound. Our approach:
  - We define a family of convex polytopes.
  - We show that the sub-family with diameter < 1 has $n^{\Omega(n)}$ connected components.
  - We apply Ben-Or's bound.

Polytopes $P(\bar{\beta})$

- The family of polytopes is parametrized by $\bar{\beta} \in \mathbb{R}^{2n-1}$.
- When n is fixed, only the $2n-1$ blue points change with $\bar{\beta}$.



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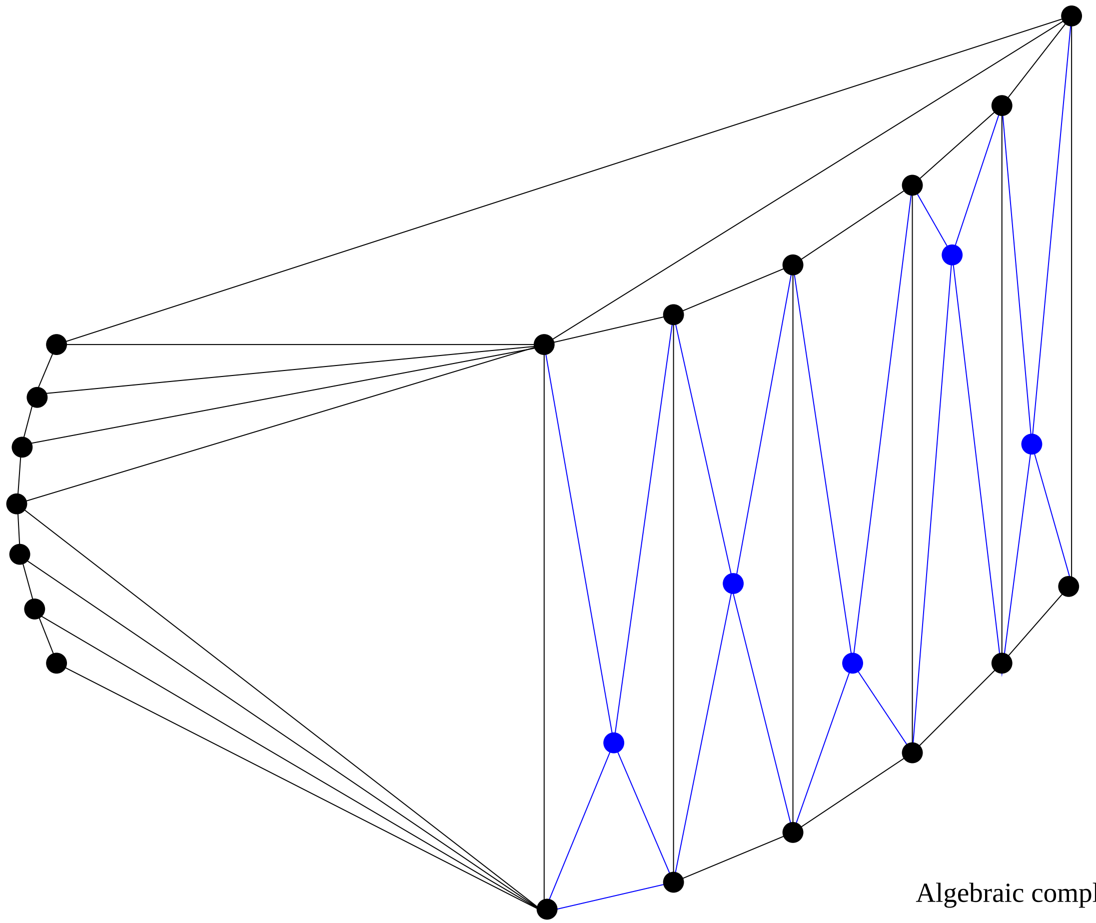
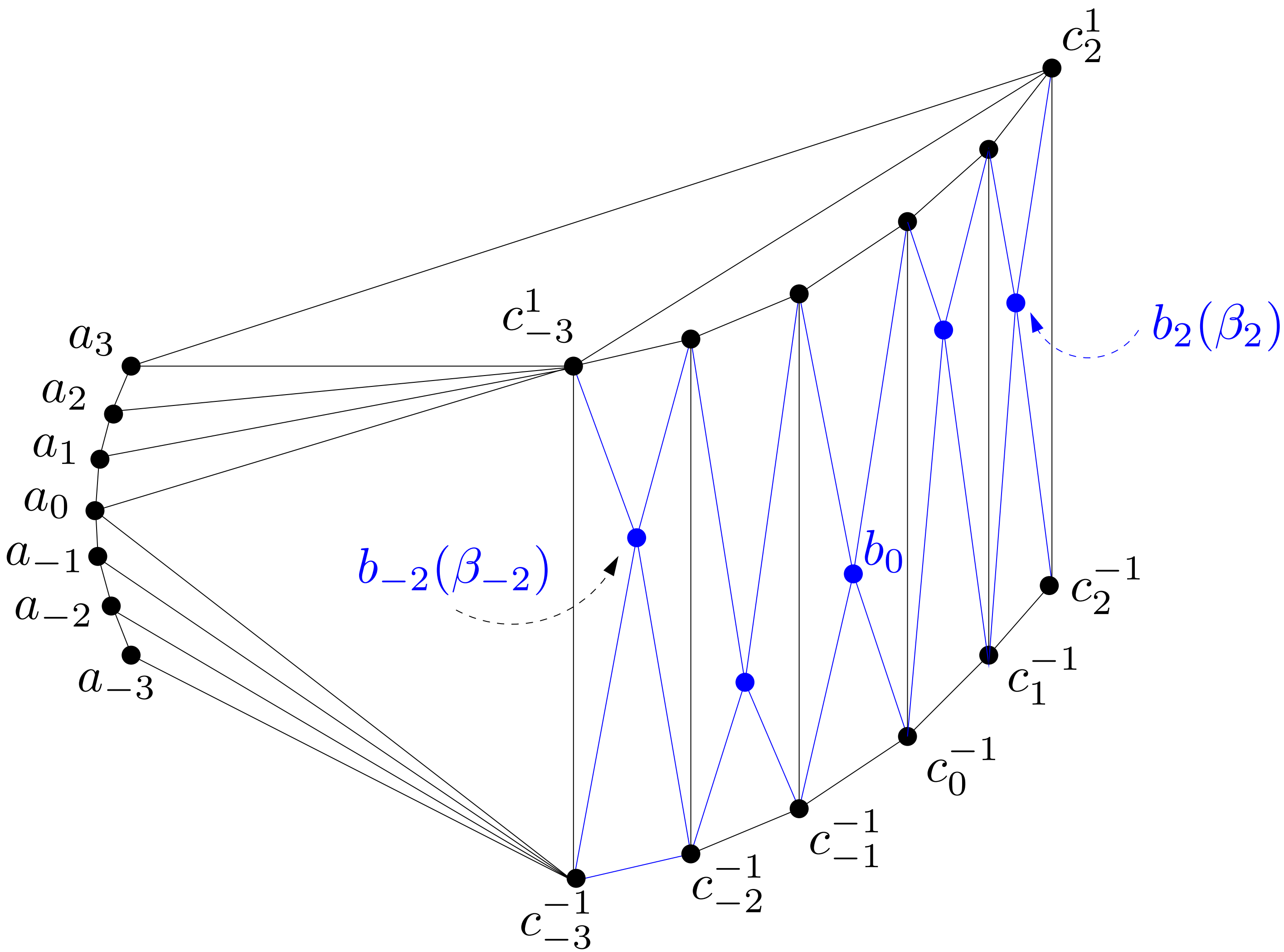


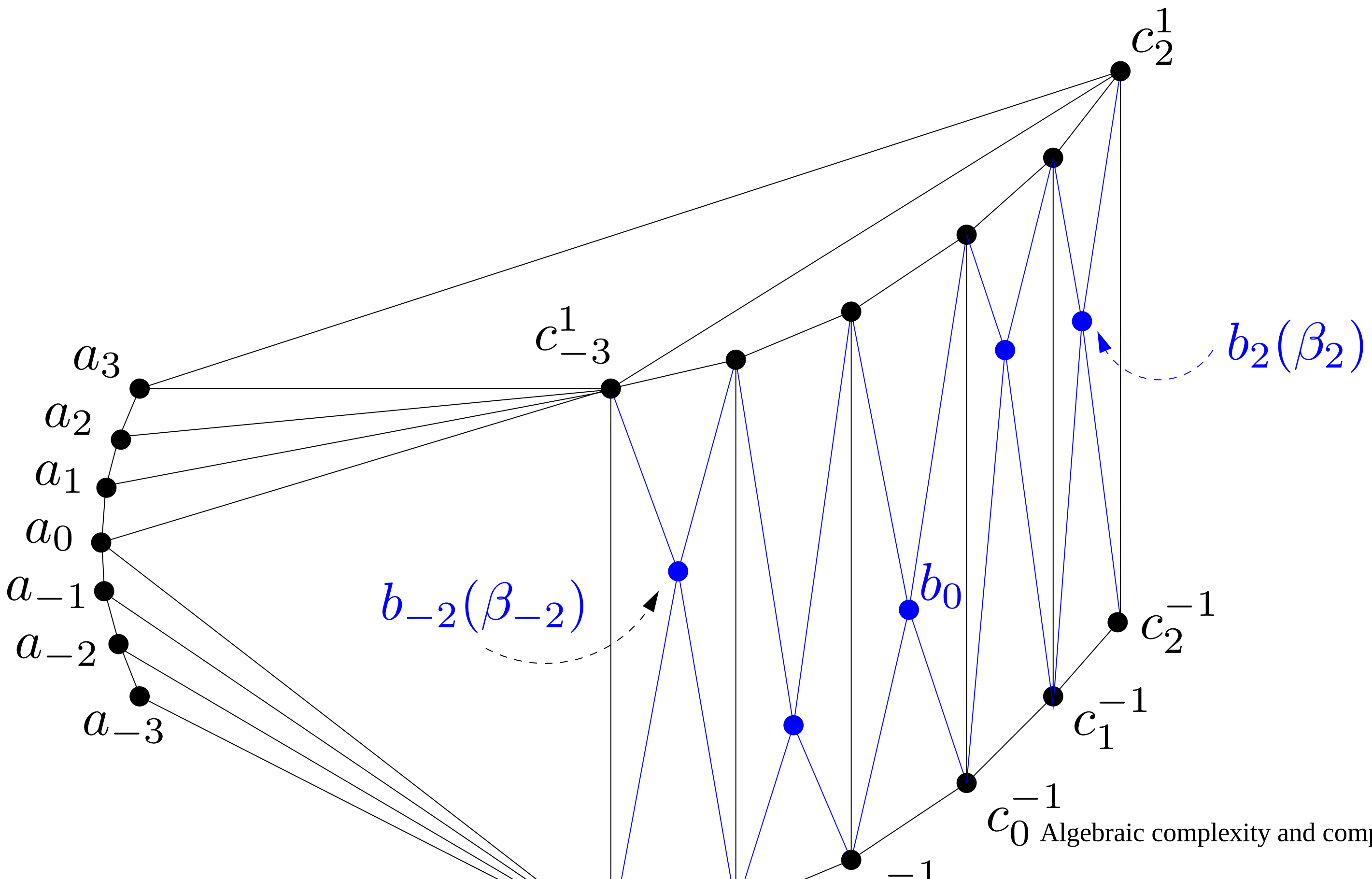
Polytopes $P(\bar{\beta})$

- Example where $n = 3$.



Properties of $P(\bar\beta)$

- The combinatorial structure of $\mathrm{CH}(A \cup B(\bar\beta) \cup C)$ is independent of $\bar\beta$.
- $\mathrm{diam}(A \cup B(\bar\beta) \cup C) = \mathrm{dist}(A, B(\bar\beta))$.

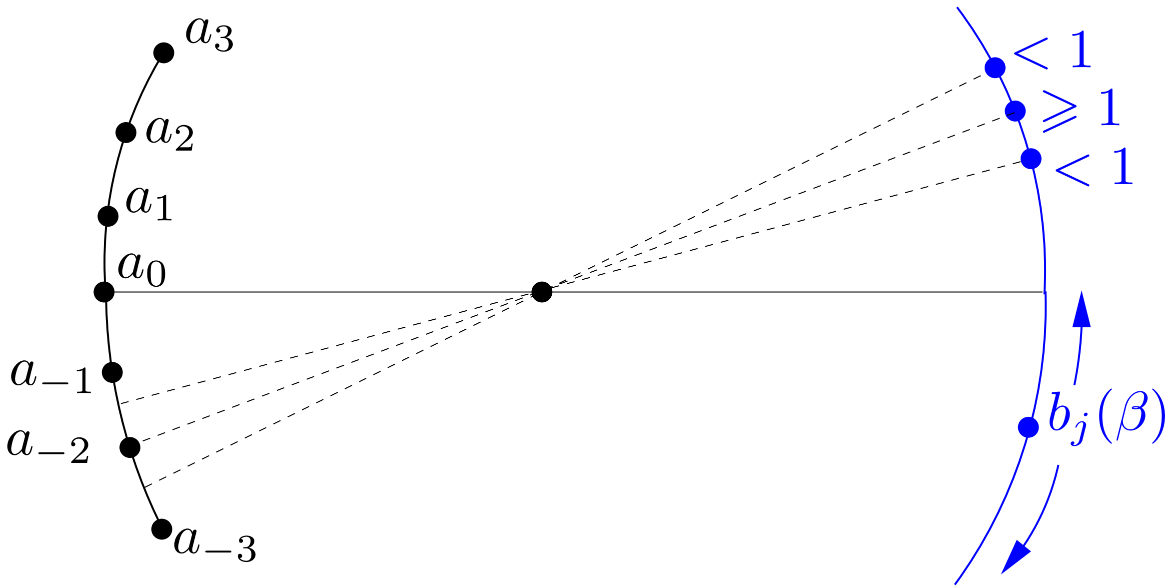


Properties of $P(\bar{\beta})$

- The set

$$\{b_j(\beta) \mid \beta \in [-\alpha, \alpha] \text{ and } \mathrm{dist}(A, \{b_j(\beta)\}) < 1\}$$

has at least $2n$ connected components.



Proof outline

- Definitions:

$$
\begin{aligned}
\mathcal{S}_n &= \{(\bar{a}, \bar{b}(\bar{\beta}), \bar{c}) \mid \bar{\beta} \in [-\alpha, \alpha]^{2n-1}\} \\
\mathcal{E}_n &= \{(\bar{a}, \bar{b}(\bar{\beta}), \bar{c}) \mid \bar{\beta} \in [-\alpha, \alpha]^{2n-1} \text{and } \mathrm{diam}(P(\bar{\beta})) < 1\}
\end{aligned}
$$

- Restriction to $\mathcal{S}_n$ is easy: the set $\mathcal{S}_n$ can be decided by an ACT with depth $O(n)$.
- Deciding $\mathcal{E}_n$ over $\mathcal{S}_n$ is hard: $\mathcal{E}_n$ has at least $(2n)^{2n-1}$ connected components. Apply Ben-Or's bound.

Randomized computation trees

- A RCT is a set of trees (T_i) with probability vector (p_i) ($p_i \geqslant 0$, $\sum_{i \in I} p_i = 1$)
- complexity = maximum depth of all T_i
- (T_i) decides S if
  - If $x \in S$: $\mathbb{P}[T_i \text{ accepts } x] > 2/3$
  - If $x \notin S$: $\mathbb{P}[T_i \text{ accepts } x] < 1/3$

Randomization can help

- Given $(x_1, \ldots, x_n, y_1, \ldots y_n)$, decide if

$$\begin{cases} y_1 = x_1 + \ldots + x_n \\ y_2 = \sum_{i<j} x_i x_j \\ \ldots \\ y_n = x_1 x_2 \ldots x_n \end{cases}$$

- $\Omega(n \log n)$ deterministic lower bound
- $O(n)$ randomized algorithm: take $\xi \sim U(\{1, 2, \ldots, 3n\})$ and check if

$$\xi^n + \sum_{i=0}^{n-1} (-1)^i y_{n-i} \xi^i = \prod_{i=1}^{n} (\xi - x_i).$$

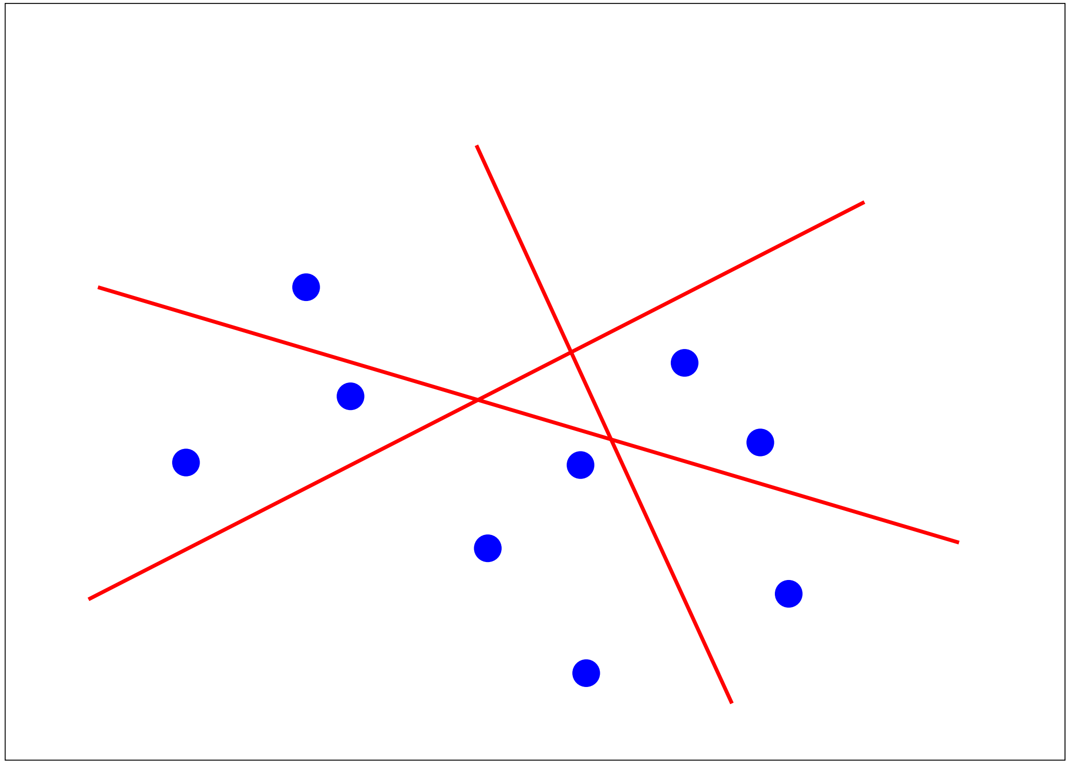
Randomized computation trees

- Question: $\Omega(n \log n)$ lower bound on diameter of 3D polytope in the RCT model?
- $\Omega(n \log n)$ randomized lower bound for Element Distinctness (Grigoriev)

Diameter is harder than Hopcroft's problem

Hopcroft's problem

- P is a set of n points in $\mathbb{R}^2$.
- L is a set of n lines in $\mathbb{R}^2$.
- Problem: decide whether $\exists (p, \ell) \in P \times L : p \in \ell$.



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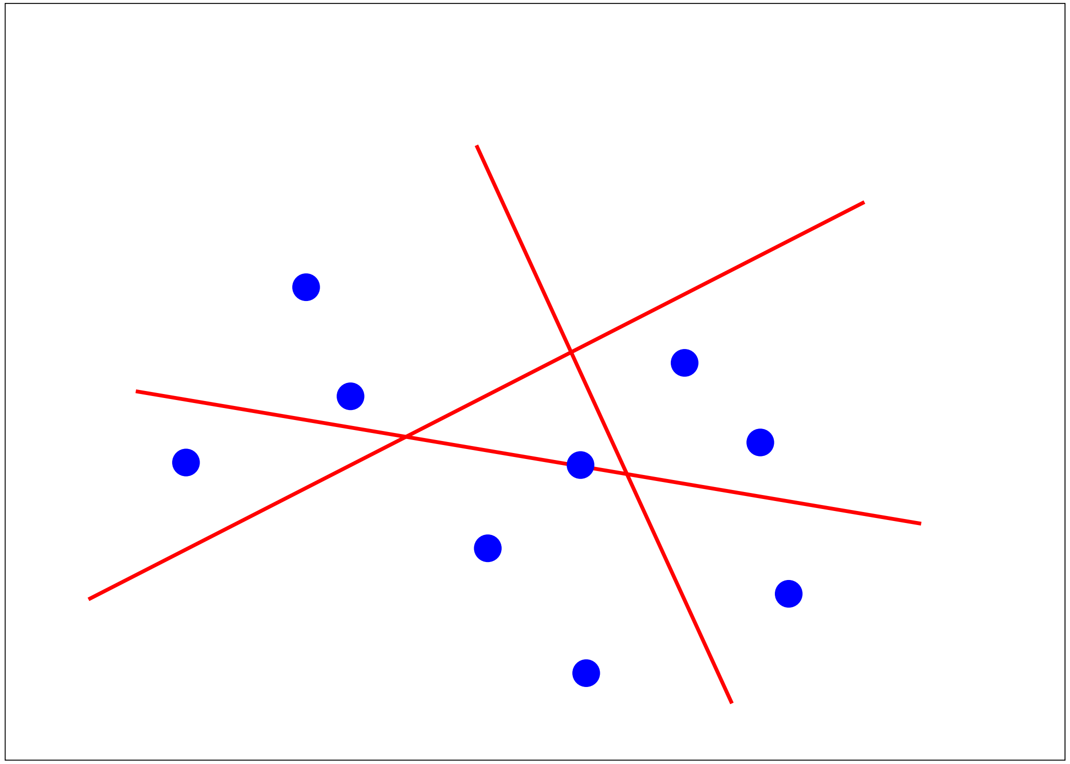

Simple upper bound for Hopcroft's problem

- Naive algorithm in time $O(n^2)$.
- Improved upper bound:
  - Divide the n lines in $\sqrt{n}$ packets of size $\sqrt{n}$;
  - For each packet in turn, compute the arrangement of lines and look for point-line incidence. This takes time
$$(\sqrt{n})^2 \log \sqrt{n} = \Theta(n \log n).$$
- This gives an algorithm in time $O(n^{3/2} \log n)$.

Complexity of Hopcroft's problem

- An $o(n^{4/3} \log n)$ algorithm is known. (Matoušek).
  - Based on highly efficient point location techniques.
- No $O(n^{4/3})$ algorithm is known.
- Erickson gave an $\Omega(n^{4/3})$ lower bound in a *weaker model*.
  - Partitioning algorithms, based on a divide-and-conquer approach.

From Hopcroft's problem to Diameter

- We give a linear-time reduction from Hopcroft's problem to the diameter problem in $\mathbb{R}^7$.
  - Known upper bound: $n^{1.6} \log^{O(1)} n$.
- We first give a reduction to the *red-blue diameter* problem in $\mathbb{R}^6$: compute $\mathrm{diam}(E, F)$ when E and F are n-point sets in $\mathbb{R}^6$.

The reduction

- $\theta(x, y, z) := \dfrac{1}{x^2 + y^2 + z^2}(x^2, y^2, z^2, \sqrt{2}xy, \sqrt{2}yz, \sqrt{2}zx)$.
- Note that $\|\theta(x, y, z)\| = 1$.
- For $1 \leqslant i \leqslant n$
  - $p_i = (x_i, y_i, 1)$
  - $\ell_i = (u_i, v_i, w_i)$ is the line $\ell_i : u_i x + v_i y + w_i = 0$.
- Let $p'_i := \theta(p_i)$ and $\ell'_j = \theta(\ell_j)$.
- We get

$$
\begin{aligned}
\|p'_i - \ell'_j\|^2 &= \|p'_i\|^2 + \|\ell'_j\|^2 - 2\langle p'_i, \ell'_j\rangle \\
&= 2 - 2\frac{\langle p_i, \ell_j\rangle^2}{\|p_i\|^2\|\ell_j\|^2}
\end{aligned}
$$

The reduction

- Note that $p_i \in \ell_j$ iff $\langle p_i, \ell_j \rangle = 0$.
- Thus, there exists i, j such that $p_i \in \ell_j$ if and only if $\mathrm{diam}(\theta(P), \theta(L)) = 2$.
- $\theta(P)$ and $\theta(L)$ are n-point sets in $\mathbb{R}^6$.
- Similarly, we can get a reduction from Hopcroft's problem to the diameter problem in $\mathbb{R}^7$, using this linearization:

$$\tilde{\theta}(x, y, z) := \left(\frac{1}{x^2 + y^2 + z^2} (x^2, y^2, z^2, \sqrt{2}xy, \sqrt{2}yz, \sqrt{2}zx), \pm 1 \right)$$

Related work

- Erickson gave reduction from Hopfcroft problem to other computational geometry problems.
  - Halfspace emptyness in $\mathbb{R}^5$,
  - Ray shooting in polyhedral terrains

  are harder than Hopcroft's problem.
- The red-blue diameter in $\mathbb{R}^4$ can be computed in $O(n^{4/3}\mathrm{polylog}\ n)$ (Matoušek and Scharzkopf).
  - Question: is there a reduction from Hopcroft's problem to red-blue diameter in $\mathbb{R}^4$?