Growth processes and balanced and/or trees

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Outline: Growth processes

- A single connector:
  - Majority function: Valiant (84)
  - Extensions: Boppana (85), Servedio (99)
  - Classification: Brodsky, Pippenger (05)
- Two connectors: $\wedge$ and $\vee$
  - Limit distribution
  - Convergence speed

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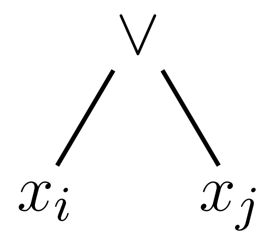

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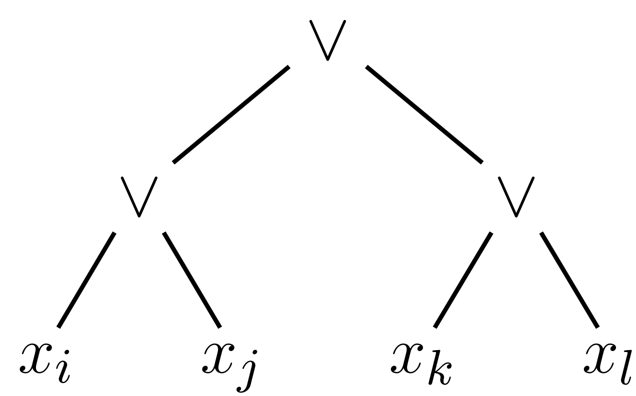

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Distributions on Boolean functions $\tilde{\pi}_n$:

$$f_i \ :\ \{0,1\}^k \rightarrow \{0,1\} \qquad f_{i,j} \ :\ \{0,1\}^k \rightarrow \{0,1\}$$
$$x \rightarrow x_i \qquad x \rightarrow x_i \vee x_j$$

$$\tilde{\pi}_0(f_i) = \frac{1}{k}$$
$$\tilde{\pi}_1(f_i) = \frac{1}{k^2} \text{ and } \tilde{\pi}_1(f_{i,j}) = \frac{2}{k^2}$$
$$\dots$$

Does the sequence $(\tilde{\pi}_n)_{n\in\mathbb{N}}$ have a limit?

If the limit exists, is it a probability distribution?

Another point of view

Large and/or trees conditioned by the fact that all leaves are at the same level.

Previous results

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- *Theorem:* Let $n_0 = O(\log k)$. It is the smallest value for this growth process such that $\forall n > n_0,\ \pi_n(Maj) \geqslant 1/2$.
- Gupta and Mahajan (97) have improved n_0 (but it is steal $O(\log k)$) by taking another growth process.

Threshold functions

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- 99: Servedio extends the results to
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Moreover they studied the speed convergence of the growth processes.

What does happen if we allow several connectors ?

Which distribution on the set of connectors ?

Balanced and/or trees

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- Let $T_\theta^{\alpha_1,\ldots,\alpha_k}$ be the *weighted θ-threshold function*:

$$T_\theta^{\alpha_1,\ldots,\alpha_k}(x) = 1 \iff \alpha_1 x_1 + \cdots + \alpha_k x_k > \theta.$$

Results

Let $p \in [0,1]$ be the probability of $\wedge$ (1-p the one of $\vee$). The growth process $\mathcal{G}_p(\pi_0)$ admits a limit probability distribution π such that

- $\pi(x_1 \wedge \cdots \wedge x_k) = 1$ if $p > 0.5$.
- $\pi(x_1 \vee \cdots \vee x_k) = 1$ if $p < 0.5$.
- If $p = 0.5$, π is the following probability distribution $D^{\alpha_1,\ldots,\alpha_k}$.

Distribution $D^{\alpha_1,\ldots,\alpha_k}$

When θ runs over $[0,1]$, we get a set of parallel hyperplanes:

$$\alpha_1 x_1 + \cdots + \alpha_k x_k = \theta.$$

Let $\theta_0, \theta_1, \ldots, \theta_r$ be the increasing sequence such that

$$\forall i \in \{0,\ldots,r\},\ \exists\, x = (x_1,\ldots,x_k) \mid \alpha_1 x_1 + \cdots + \alpha_k x_k = \theta_i.$$

Under $D^{\alpha_1,\ldots,\alpha_k}$, for each $i \in \{0,\ldots,r-1\}$
the probability of the weighted θ_i-threshold function is

$$\theta_{i+1} - \theta_i.$$

Example with 2 variables

Let $\pi_c(\wedge) = \pi_c(\vee) = 0.5$.
Let $H_0 = \{x_1, x_2\}$, $\pi_0(x_1) = 1/5$ and $\pi_0(x_2) = 4/5$.

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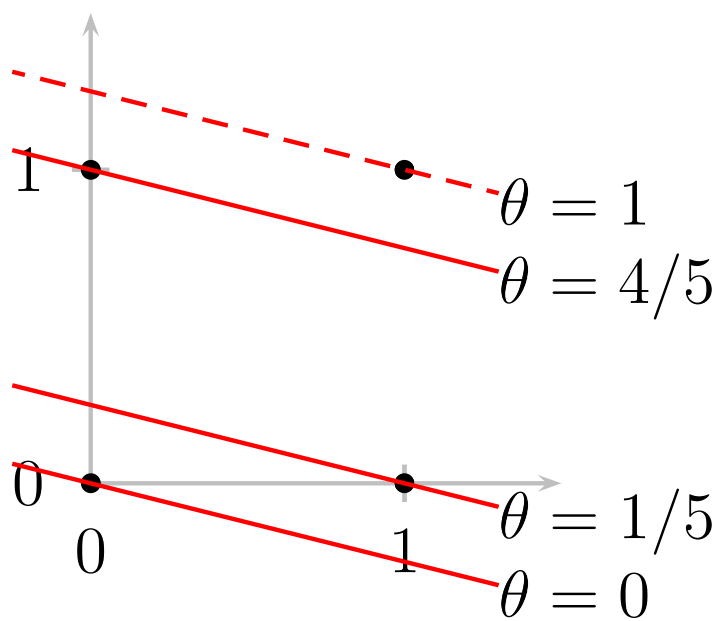

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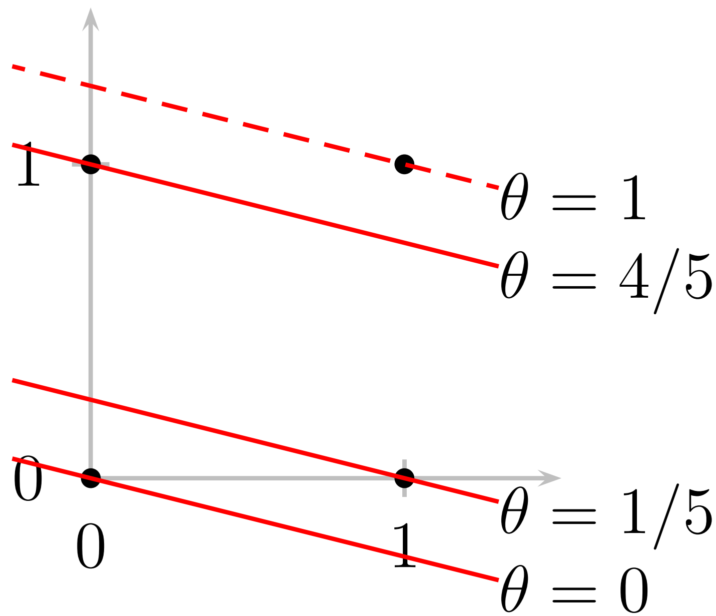

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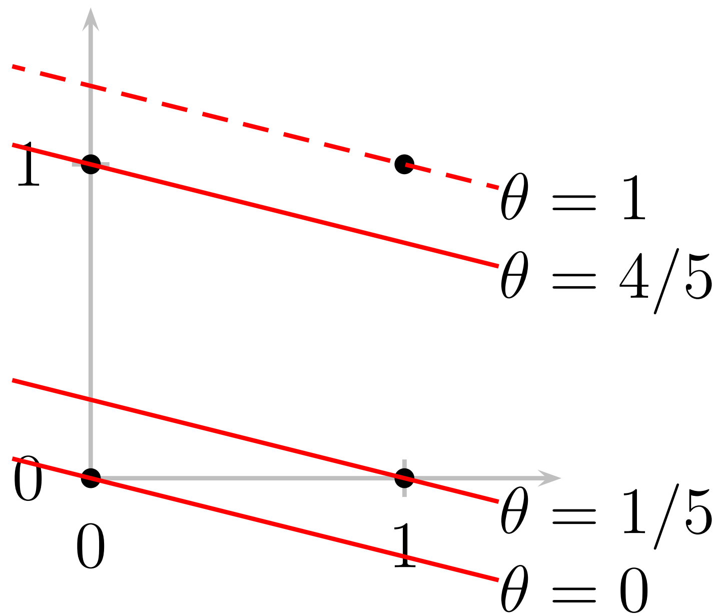

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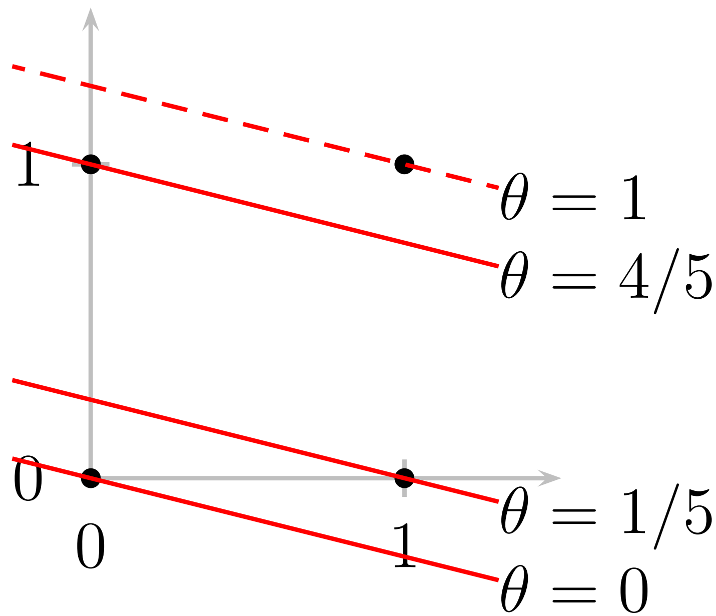

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- Limit probability of $T_{1/5}^{1/5,4/5}$ (i.e. $x \to x_2$): $3/5$.
- Limit probability of $T_{4/5}^{1/5,4/5}$ (i.e. $x \to x_1 \wedge x_2$): $1/5$.

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- Study by recurrence of the limit proba. of each $T_\theta^{\alpha_1,\ldots,\alpha_k}$.

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- Adding the probabilities of the repeated functions.

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The limit distribution π exists and

is uniform on both constant functions 1 and 0.

Convergence speed

Let P be the growth process. Let denote π its limit

distribution and n_ϵ the smallest value such that

$$\forall \epsilon,\ \forall n > n_\epsilon,\ max_{f \in \mathcal{B}_k} |\pi_n(f) - \pi(f)| \leqslant \epsilon.$$

If all assignments have different weights,

then $n_\epsilon \sim k \log(1/\epsilon)$, otherwise $n_\epsilon \sim k/\epsilon$.

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- Balancedness really modifies the limit distribution.
- What does happen if we change the set of connectors?
- By changing the connectors, could we obtain a better convergence speed?